The order of non-ordinary points under a blow-up

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Abstract : Let $\pi : X \to C$ be a relatively minimal genus one fibration and assume that $R^1\pi_*\mathcal{O}_X$ is locally free. Let $b : \widetilde{X} \to X$ be a blow-up at a smooth point $x \in \pi^{-1}(y)$. We will show that the order of non-ordinary point $y \in C$ is equal to the length of the higher direct image of the cokernel of the relative Frobenius morphism of $\pi \circ b$ at y.

Key words : genus one fibration, non-ordinary point, Frobenius morphism

1. Introduction

Let p be a prime number. The Hasse invariant is one of the most fundamental invariant of elliptic curves in characteristic p. This invariant is defined by the number of p-torsion points and has only 0 or 1. An elliptic curve E with the Hasse invariant 0 is called supersingular, otherwise ordinary. There are several characterizations of the Hasse invariant (cf. [5, Chapter V, Section 3]. Especially we want to mention that this invariant can be also defined by the Frobenius action $F_E^* : H^1(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E)$ (cf. [3, Chapter IV, Section 4]).

Motivated by the importance of the Hasse invariant, we will study an invariant of families of elliptic curves which is called the *non-ordinary points* and their *order* in this note. These notions was firstly defined by [4]. In particular, we will study these notions in genus one fibrations $\pi : X \to C$.

Let us recall these definitions briefly. Let $\pi : X \to C$ be a relatively minimal genus one fibration without wild fibers (See Definition 2.6), and $F_C : C \to C$ the absolute Frobenius morphism. The fiber product $X^{(p)} := X \times_{(F_C,\pi)} C$ induces the relative Frobenius morphism $F_{X/C} : X \to X^{(p)}$ and the projection $X^{(p)} \to C$ is denoted by $\pi^{(p)}$. If we take the higher direct image of $\pi^{(p)}$, we have a coherent sheaf $R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C}$, where $\mathcal{B}^1_{X^{(p)}/C}$ is the cokernel of $F^\sharp_{X/C} : \mathcal{O}_{X^{(p)}} \to (F_{X/C})_*\mathcal{O}_X$. If this is a nonzero torsion sheaf, we define a non-ordinary point as a closed point in $\mathrm{Supp}(R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C})$ and the order of non-ordinary points as the length of $\mathcal{O}_{C,y}$-module $(R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C})_y$ at non-ordinary point $y \in C$.

In this note, we are interested in how the order of non-ordinary points changes under a blow-up at a smooth point $b : \widetilde{X} \to X$. That is, our main result is as follows.

Main Theorem 1.1. (= Corollary 3.5) *Let $\pi : X \to C$ be a relatively minimal genus one fibration and assume that $R^1\pi_*\mathcal{O}_X$ is locally free and $R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C}$ is a nonzero torsion sheaf. Let $b : \widetilde{X} \to X$ be a blow-up at a smooth point $x \in F_y \subset X$ where $F_y := \pi^{-1}(y)$ is the fiber of π at non-ordinary point $y \in C$. Then* $\mathrm{length}_{\mathcal{O}_{C,y}}(R^1(\pi^{(p)} \circ b^{(p)})_*\mathcal{B}^1_{\widetilde{X}^{(p)}/C})_y$ *is equal to the order of non-ordinary point $y \in C$.*

This note is organized as follows. We prepare some notations and definitions in Section 2. Especially we define non-ordinary points and their order in this section. In Section 3, we claim some lemmas to prove our result in Theorem 3.4. Finally, we state our main theorem as a corollary of Theorem 3.4.

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2. Preliminaries

Throughout this note, p is a prime number greater than 3. A field k is algebraically closed of characteristic p and variety means irreducible, reduced, separated and of finite type scheme over k. We define a curve or surface as a one or two-dimensional variety respectively. A scheme X is called k-scheme if X has a structure morphism $X \to \mathrm{Spec}k$ and a morphism $f : X \to Y$ is called k-morphism if f is compatible with the structure morphisms X and Y.

In this section, we will review some basic definitions and notions which we treat repeatedly.

Definition 2.1. Let X be a scheme over k. *The absolute Frobenius morphism*

$$F_X : X \to X$$

is the identity map on the underlying topological space X and the morphism between sheaves

$$\begin{array}{rcl} F_X^{\sharp} : \mathcal{O}_X & \to & (F_X)_*\mathcal{O}_X \\ f & \mapsto & f^p \end{array}$$

is defined by its p-th powers.

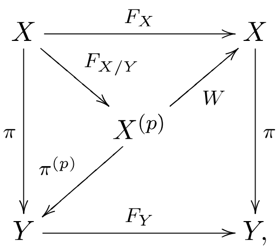
Remark 2.2. The absolute Frobenius morphism is not a k-morphism in general.

Let X and Y be k-schemes. If we have a following commutative diagram;

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[F_Y]{} & Y, \end{array}$$

taking fiber product $X^{(p)} := X \times_{(F_Y,\pi)} Y$, the relative Frobenius morphism is defined by the following.

Definition 2.3. Let $\pi : X \to Y$ be a morphism between k-schemes. *The relative Frobenius morphism* $F_{X/Y} : X \to X^{(p)} := X \times_{(F_Y,\pi)} Y$ is the universal morphism of the following diagram;



where $\pi^{(p)} : X^{(p)} \to Y$ and $W : X^{(p)} \to X$ are projections respectively.

Genus one fibrations play central roles in this note. These are called sometimes *elliptic fibrations*. However in this note, we do not care about sections of these fibrations. We recall some definitions and their fibers briefly. We list [1, Section 7] as a reference.

Definition 2.4. Let X be a smooth projective surface over k and C a smooth projective curve over k. A surjective morphism $\pi : X \to C$ is called a *genus one fibration* if the following conditions satisfy;
(i) $\pi_*\mathcal{O}_X = \mathcal{O}_C$,
(ii) The general fibers of π are a smooth curves of genus one.

Moreover a genus one fibration $\pi : X \to C$ is called *relatively minimal* if every fiber has no (-1)-curves.

Remark 2.5. Since C is a one-dimensional smooth scheme, the surjectivity of π implies the flatness of π and the connectedness of the fibers is followed by the condition (i) [3, Chapter III, Corollary 11.3].

Let $\pi : X \to C$ be a relatively minimal genus one fibration and F_y a fiber of π at a closed point $y \in C$. Since the base curve C is a smooth curve, the coherent sheaf $R^1\pi_*\mathcal{O}_X$ decomposes as follows;

$$R^1\pi_*\mathcal{O}_X = \mathcal{E} \oplus \mathcal{T},$$

where $\mathcal{E}$ is the locally free part and $\mathcal{T}$ is the torsion part of $R^1\pi_*\mathcal{O}_X$ respectively. By the base change theorem, there is an isomorphism

$$R^1\pi_*\mathcal{O}_X \otimes k(y) \cong H^1(F_y, \mathcal{O}_{F_y}),$$

where $k(y)$ is the residue field at $y \in C$. Since the genus of the general fiber is one, $\mathcal{E}$ is a line bundle $\mathcal{L}$.

Definition 2.6. A fiber F_y is called *wild* if $y \in \mathrm{Supp}\mathcal{T}$. If the fiber is not wild, it is called *tame*.

Remark 2.7. There are some examples of wild fibers in low characteristic in [2].

Let $\pi : X \to C$ be a relatively minimal genus one fibration and assume that $\mathcal{L} := R^1\pi_*\mathcal{O}_X$ is locally free. Let $\mathcal{B}^1_{X^{(p)}/C}$ be the cokernel of the relative Frobenius morphism of π. Taking the higher direct image of $\pi^{(p)}$, there is an exact sequence;

$$R^1\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \xrightarrow{F^*_{X/C}} R^1\pi^{(p)}_*((F_{X/C})_*\mathcal{O}_X) \to R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C} \to 0. \tag{1}$$

Since F_C is a flat morphism, $R^1\pi^{(p)}_*\mathcal{O}_{X^{(p)}}$ is isomorphic to $F_C^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$ by flat base extension [3, Chapter III, Proposition 9.3]. Since $F_{X/C}$ is a finite morphism, $R^1\pi^{(p)}_*((F_{X/C})_*\mathcal{O}_X)$ is also isomorphic to $R^1\pi_*\mathcal{O}_X$ by the Leray spectral sequence. Thus, we can regard the map $F^*_{X/C}$ in the exact sequence (1) as the global section of $\mathcal{L}^{\otimes(1-p)}$.

Let $y \in C$ be a closed point and $k(y)$ the residue field at y. The above exact sequence (1) also induces the following;

$$R^1\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \otimes k(y) \xrightarrow{F^*_{X/C}\otimes \mathrm{id}_{k(y)}} R^1\pi^{(p)}_*((F_{X/C})_*\mathcal{O}_X) \otimes k(y) \to R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C} \otimes k(y) \to 0,$$

By the base change theorem, this is equivalent to the following;

$$H^1(F_y^{(p)}, \mathcal{O}_{F_y^{(p)}}) \xrightarrow{F^*_{F_y}} H^1(F_y, \mathcal{O}_{F_y}) \to R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C} \otimes k(y) \to 0,$$

where the map $F^*_{F_y}$ is the Frobenius action of the fiber $F_y := \pi^{-1}(y)$. This Frobenius action $F^*_{F_y}$ is the map between one-dimensional vector spaces over $k(y)$. Since the dimension of the vector space $R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C} \otimes k(y)$ is at most one, $R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C}$ is a line bundle or torsion sheaf.

If it is a line bundle, then by the surjectivity of line bundles, $R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C}$ is isomorphic to $\mathcal{L}$. Thus, $F^*_{X/C}$ is zero map and $F^*_{F_y}$ is also zero for all closed points $y \in C$.

If $R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C}$ is a torsion sheaf, we have a notion of non-ordinary points as follows.

Definition 2.8. [4, Definition 4.1] Let $\pi : X \to C$ be a relatively minimal genus one fibration and assume that $R^1\pi_*\mathcal{O}_X$ is locally free and $R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C}$ is a torsion sheaf. A closed point $y \in C$ is called *non-ordinary point* if $y \in \mathrm{Supp}(R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C})$. The *order* of non-ordinary point is defined by the length of $(R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C})_y$ as $\mathcal{O}_{C,y}$-module.

Remark 2.9. Suppose that $R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C}$ is nonzero torsion sheaf. Since $F^*_{X/C} : \mathcal{L}^{\otimes p} \to \mathcal{L}$ is a nonzero map between line bundles on a smooth curve C, this is injective. Thus we have the following exact sequence;

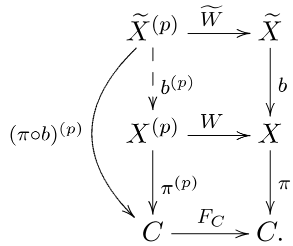
$$0 \to \mathcal{L}^{\otimes p} \xrightarrow{F^*_{X/C}} \mathcal{L} \to R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C} \to 0. \tag{2}$$

Note that the degree of a torsion sheaf $\mathcal{T}$ on a smooth projective curve is defined by $\deg\mathcal{T} = \sum_{x\in C} \mathrm{length}_{\mathcal{O}_{C,x}}(\mathcal{T}_x)$. In our exact sequence (2), $\deg(R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C})$ is equal to $(1-p)\cdot\deg\mathcal{L}$ by [3, Chapter II, Exercise 6.12].

3. Main result

Let $\pi : X \to C$ be a relatively minimal genus one fibration and assume that $\mathcal{L} := R^1\pi_*\mathcal{O}_X$ is locally free and $R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C}$ is a torsion sheaf. In the previous section, we defined the order of non-ordinary points. We are interested in how this order changes under blow-ups. To state our result more precisely, we prepare some notations.

Let $b : \widetilde{X} \to X$ be a blow-up at a smooth closed point $x \in F_y \subset X$ where F_y is the fiber of π at a closed point $y \in C$. The base change of $\widetilde{X}$ and $\pi \circ b$ by F_C is denoted by $\widetilde{X}^{(p)}$ and $(\pi\circ b)^{(p)}$ respectively. Let $\widetilde{W} : \widetilde{X}^{(p)} \to \widetilde{X}$ be the projection. By the universality of fiber products, there is the morphism $b^{(p)} : \widetilde{X}^{(p)} \to X^{(p)}$. Thus $(\pi\circ b)^{(p)} = \pi^{(p)} \circ b^{(p)}$ as follows;



Lemma 3.1. *The higher direct image sheaf* $R^1(\pi\circ b)_*\mathcal{O}_{\widetilde{X}}$ *is isomorphic to* $\mathcal{L} = R^1\pi_*\mathcal{O}_X$ *and* $R^1(\pi\circ b)_*^{(p)}\mathcal{O}_{\widetilde{X}^{(p)}}$ *is isomorphic to* $\mathcal{L}^{\otimes p}$.

Proof. This is due to the Leray spectral sequence. Let $E_2^{r,s}$ be $\mathcal{O}_C$-modules $R^r\pi_*(R^s b_*\mathcal{O}_{\widetilde{X}})$ $(r, s \in \mathbb{Z}_{\geq 0})$ and E^{r+s} be $\mathcal{O}_C$-modules $R^{r+s}(\pi\circ b)_*\mathcal{O}_{\widetilde{X}}$. By the Leray spectral sequence $E_2^{r,s} \Rightarrow E^{r+s}$, there is the exact sequence;

$$0 \to R^1\pi_*(b_*\mathcal{O}_X) \to R^1(\pi\circ b)_*\mathcal{O}_{\widetilde{X}} \to \pi_*(R^1b_*\mathcal{O}_{\widetilde{X}}).$$

Since b is a blow-up at a smooth point, $R^1b_*\mathcal{O}_{\widetilde{X}}$ is zero and $b_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ [3, Chapter IV, Proposition 3.4]. $R^1(\pi\circ b)_*\mathcal{O}_{\widetilde{X}}$ is isomorphic to $R^1\pi_*(b_*\mathcal{O}_{\widetilde{X}}) \cong R^1\pi_*\mathcal{O}_X = \mathcal{L}$.

Since F_C is flat, by flat base extension, we have the following isomorphisms;

$$\begin{aligned} R^1(\pi\circ b)_*^{(p)}\mathcal{O}_{\widetilde{X}^{(p)}} &\cong R^1(\pi\circ b)_*^{(p)}\widetilde{W}^*\mathcal{O}_{\widetilde{X}} \\ &\cong F_C^*R^1(\pi\circ b)_*\mathcal{O}_{\widetilde{X}} \\ &\cong F_C^*R^1\pi_*\mathcal{O}_X \\ &\cong \mathcal{L}^{\otimes p}. \end{aligned}$$

□

Lemma 3.2. *Let $y' \in C$ be a closed point which is not y. Then there is an isomorphism of $\mathcal{O}_{C,y'}$-modules*

$$(R^1\pi_*^{(p)}(b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C}))_{y'} \cong (R^1(\pi^{(p)} \circ b^{(p)})_*\mathcal{B}^1_{\widetilde{X}^{(p)}/C}))_{y'}$$

where $\mathcal{B}^1_{\widetilde{X}^{(p)}/C}$ is the cokernel of $F^\sharp_{\widetilde{X}/C} : \mathcal{O}_{\widetilde{X}^{(p)}} \to (F_{\widetilde{X}/C})_*\mathcal{O}_{\widetilde{X}}$.

Proof. We have an injective map by the Leray spectral sequence as follows;

$$0 \to R^1\pi_*^{(p)}(b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C}) \to R^1(\pi^{(p)} \circ b^{(p)})_*\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \to \pi_*^{(p)}R^1b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \to 0.$$

Thus, it is enough to show that $(\pi_*^{(p)}R^1b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C})_{y'}$ is zero.

Taking an affine open neighbourhood $U = \mathrm{Spec}A$ of y' not containing y, we may assume that C is $\mathrm{Spec}A$. U' and U'' denote $\pi^{-1}(U)$ and $b^{-1}(U')$ respectively. By the definition of higher direct image, we have the following;

$$(\pi_*^{(p)}R^1b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C}) \mid_U \cong H^0(U'^{(p)}, R^1b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \mid_{U'^{(p)}})^{\sim},$$

where $U'^{(p)}$ is the base change of U' by F_U which is the absolute Frobenius morphism of U. Since $R^1b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \mid_{U'^{(p)}} = R^1(b^{(p)} \mid_{U''^{(p)}})_*(\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \mid_{U''^{(p)}})$, where $U''^{(p)}$ is the base change of U'' by F_U and $b^{(p)}$ is the isomorphism on $U''^{(p)}$, then the sheaf $R^1b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \mid_{U'^{(p)}}$ vanishes. The stalk $(\pi_*^{(p)}R^1b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C})_{y'}$ is also zero. □

Lemma 3.3. *Let $y' \in C$ be a closed point as above. We have the following isomorphism of $\mathcal{O}_{C,y'}$-modules;*

$$(R^1\pi_*^{(p)}(b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C}))_{y'} \cong (R^1\pi_*^{(p)}\mathcal{B}^1_{X^{(p)}/C})_{y'}.$$

Proof. Taking an affine open neighbourhood of $y' \in C$ in the previous lemma similarly, it is enough to show that the isomorphism on $U = \mathrm{Spec}A$. The sheaf $R^1\pi_*^{(p)}(b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C})$ can be described as follows;

$$R^1\pi_*^{(p)}(b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C}) \mid_U \cong H^1(U'^{(p)}, b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \mid_{U'^{(p)}})^{\sim}. \quad (3)$$

Since $F_{X/C}$ and $F_{\widetilde{X}/C}$ are defined as the universal morphisms, we have the following commutative diagram;

$$\begin{array}{ccccc}
\widetilde{X} & \xrightarrow{F_{\widetilde{X}/C}} & \widetilde{X}^{(p)} & \xrightarrow{\widetilde{W}} & \widetilde{X} \\
\downarrow b & & \downarrow b^{(p)} & & \downarrow b \\
X & \xrightarrow[F_{X/C}]{} & X^{(p)} & \xrightarrow[W]{} & X.
\end{array}$$

This diagram induces a morphism between exact sequences as follows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{X^{(p)}} & \longrightarrow & (F_{X/C})_*\mathcal{O}_X & \longrightarrow & \mathcal{B}^1_{X^{(p)}/C} & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & & & \\
0 & \longrightarrow & b_*^{(p)}\mathcal{O}_{\widetilde{X}^{(p)}} & \longrightarrow & b_*^{(p)}((F_{\widetilde{X}/C})_*\mathcal{O}_{\widetilde{X}}) & \longrightarrow & b_*^{(p)}\mathcal{B}^1_{\widetilde{X}^{(p)}/C} & \longrightarrow & R^1b_*^{(p)}\mathcal{O}_{\widetilde{X}^{(p)}}.
\end{array}$$

If we consider these exact sequences on $U'^{(p)}$, then the second row is a short exact sequence. α is an isomorphism on $U'^{(p)}$ since $b^{(p)}$ is the isomorphism on $U''^{(p)}$. Moreover β is also an isomorphism since $b_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$. Therefore α and β induce a morphism $\gamma : \mathcal{B}^1_{X^{(p)}/C} \mid_{U'^{(p)}} \to b^{(p)}_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C} \mid_{U'^{(p)}}$ and γ is an isomorphism on $U'^{(p)}$. So by (3),

$$\begin{aligned} H^1(U'^{(p)}, b^{(p)}_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C} \mid_{U'^{(p)}})^{\widetilde{}} &\cong H^1(U'^{(p)}, \mathcal{B}^1_{X^{(p)}/C} \mid_{U'^{(p)}})^{\widetilde{}} \\ &\cong R^1\pi^{(p)}_* \mathcal{B}^1_{X^{(p)}/C} \mid_U . \end{aligned}$$

□

Theorem 3.4. *Let $\pi : X \to C$ be a relatively minimal genus one fibration and assume that $\mathcal{L} := R^1\pi_*\mathcal{O}_X$ is locally free and $R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C}$ is a nonzero torsion sheaf. Let F_y be a closed fiber of π at $y \in C$ and x a closed point in $F_y \subset X$. Suppose that $b : \widetilde{X} \to X$ is a blow-up at $x \in X$. Then the length of $R^1(\pi^{(p)} \circ b^{(p)})_*\mathcal{B}^1_{\widetilde{X}^{(p)}/C}$ at y is equal to that of $R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C}$ at y.*

Proof. By the definition of cokernels $\mathcal{B}^1_{\widetilde{X}^{(p)}/C}$ and $\mathcal{B}^1_{X^{(p)}/C}$, we have two short exact sequences;

$$0 \to \mathcal{O}_{\widetilde{X}^{(p)}} \to (F_{\widetilde{X}/C})_*\mathcal{O}_{\widetilde{X}} \to \mathcal{B}^1_{\widetilde{X}^{(p)}/C} \to 0 \qquad (4)$$

and

$$0 \to \mathcal{O}_{X^{(p)}} \to (F_{X/C})_*\mathcal{O}_X \to \mathcal{B}^1_{X^{(p)}/C} \to 0. \qquad (5)$$

Taking the higher direct image of $\pi^{(p)}$ to (5), we have the following exact sequence;

$$0 \to R^1\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \to R^1\pi^{(p)}_*((F_{X/C})_*\mathcal{O}_X) \to R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C} \to 0.$$

This exact sequence is equivalent to the following;

$$0 \to \mathcal{L}^{\otimes p} \to \mathcal{L} \to R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C} \to 0.$$

So the degree of $R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C}$ is $(1-p)\mathrm{deg}\mathcal{L}$ by Remark 2.9.

On the other hand, taking the higher direct image of $\pi^{(p)} \circ b^{(p)}$ to (4), we have the following exact sequence;

$$R^1(\pi^{(p)} \circ b^{(p)})_*\mathcal{O}_{\widetilde{X}^{(p)}} \xrightarrow{F^*_{\widetilde{X}/C}} R^1(\pi^{(p)} \circ b^{(p)})_*((F_{\widetilde{X}/C})_*\mathcal{O}_{\widetilde{X}}) \to R^1(\pi^{(p)} \circ b^{(p)})_*\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \to 0. \qquad (6)$$

Let $y' \in C$ be a closed point which is not $y \in C$. By the base change theorem, the above exact sequence is as follows;

$$H^1(\widetilde{F_{y'}}^{(p)}, \mathcal{O}_{\widetilde{F_{y'}}^{(p)}}) \xrightarrow{F^*_{\widetilde{F_{y'}}}} H^1(\widetilde{F_{y'}}, \mathcal{O}_{\widetilde{F_{y'}}}) \to R^1(\pi^{(p)} \circ b^{(p)})_*\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \otimes k(y') \to 0,$$

where $\widetilde{F_{y'}}$ is the fiber of $\pi \circ b$ at $y' \in C$ and $k(y')$ is the residue field at y'. Since y' is distinct from $y \in C$, the fiber $\widetilde{F_y}$ is isomorphic to $F_{y'}$ where F'_y is the fiber of π at $y' \in C$. Thus there exists a closed point $y' \in C$ such that $F^*_{\widetilde{F_{y'}}}$ is nonzero. Moreover, by Lemma 3.1 the above exact sequence (6) is equivalent to the following short exact sequence;

$$0 \to \mathcal{L}^{\otimes p} \xrightarrow{F^*_{\widetilde{X}/C}} \mathcal{L} \to R^1(\pi^{(p)} \circ b^{(p)})_*\mathcal{B}^1_{\widetilde{X}^{(p)}/C} \to 0.$$

Therefore $\deg R^1(\pi^{(p)} \circ b^{(p)})_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C}$ is $(1-p)\deg\mathcal{L}$. By Remark 2.9, the degree of torsion sheaves are following;

$$\deg R^1\pi^{(p)}_* \mathcal{B}^1_{X^{(p)}/C} = \mathrm{length}_{\mathcal{O}_{C,y}}(R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C})_y + \sum_{\substack{y' \in C \\ y' \neq y}} \mathrm{length}_{\mathcal{O}_{C,y'}}(R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C})_{y'}$$

and

$$\begin{aligned} \deg R^1(\pi^{(p)} \circ b^{(p)})_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C} &= \mathrm{length}_{\mathcal{O}_{C,y}}(R^1(\pi^{(p)} \circ b^{(p)})_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C})_y \\ &+ \sum_{\substack{y' \in C \\ y' \neq y}} \mathrm{length}_{\mathcal{O}_{C,y'}}(R^1(\pi^{(p)} \circ b^{(p)})_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C})_{y'}. \end{aligned}$$

By Lemma 3.2 and Lemma 3.3, it follows that

$$(R^1(\pi^{(p)} \circ b^{(p)})_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C})_{y'} \cong (R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C})_{y'}$$

for $y' \neq y$. This implies the following equation;

$$\mathrm{length}_{\mathcal{O}_{C,y}}(R^1(\pi^{(p)} \circ b^{(p)})_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C})_y = \mathrm{length}_{\mathcal{O}_{C,y}}(R^1\pi^{(p)}_*\mathcal{B}^1_{X^{(p)}/C})_y$$

□

From Theorem 3.4, we have the following statement as a corollary.

Corollary 3.5. *Let $y \in C$ be a non-ordinary point in Theorem 3.4. Then the length of $R^1(\pi^{(p)} \circ b^{(p)})_* \mathcal{B}^1_{\widetilde{X}^{(p)}/C}$ at y is equal to the order of non-ordinary point $y \in C$.*

Proof. This is directly followed by Theorem 3.4. □

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