Flat and projective properties in a torsion theory

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Abstract

In this paper we study the concepts of F-flatness and F-projectivity of modules, the latter is a dual of F-injectivity in O. Goldman [6]. We also study the existence of F-flat covers and F-projective covers.

1. Introduction. Let R be an associative ring with identity, and all modules are assumed to be unitary. We fix a hereditary torsion theory $(\mathfrak{T}, \mathfrak{F})$ for the category ${}_R\mathfrak{M}$ of left R-modules (see Dickson [4]). For each $M \in {}_R\mathfrak{M}$ there exists a unique submodule $M_t \in \mathfrak{T}$ such that $M/M_t \in \mathfrak{F}$. It is well known that with $\mathfrak{T}$ can be associated an idempotent topologizing filter $F(\mathfrak{T}) = \{I \subset R \mid I$ is a left ideal of R and $R/I \in \mathfrak{T}\}$.

Let $(*) : O \to M' \to M \to M'' \to O$ be an exact sequence of left R-modules. Recall that a left R-module Q is *F-injective* if, for every exact sequence $(*)$ with $M'' \in \mathfrak{T}$, the associated sequence $O \to \mathrm{Hom}_R(M'', Q) \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q) \to O$ is exact. Dually, a left R-module P is *F-projective* if, for every exact sequence $(*)$ with $M' \in \mathfrak{F}$, the associated sequence $O \to \mathrm{Hom}_R(P, M') \to \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, M'') \to O$ is exact. A right R-module B is *F-flat* if, for every exact sequence $(*)$ with $M'' \in \mathfrak{T}$, the associated sequence $O \to B \otimes_R M' \to B \otimes_R M \to B \otimes_R M'' \to O$ is exact.

We can formulate several characterizations of F-flat modules. We give some conditions that every $I \in F(\mathfrak{T})$ is F-projective. We also study the existence of F-flat covers and F-projective covers. Most of our methods are elementary, being modeled after standard techniques of Rotman [7].

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2. F-flats. For a right R-module B, we shall denote by B^* its character module $\mathrm{Hom}_Z(B, Q/Z)$, where Q/Z is the rationals mod one. Then a sequence of right R-modules $U \to V \to W$ is exact if and only if the sequence of character modules $W^* \to V^* \to U^*$ is exact. For any pair of modules $({}_RA, B_R)$, there is a natural isomorphism $(B \otimes_R A)^* \cong \mathrm{Hom}_R(A, B^*)$. Using these facts we obtain the following propositions.

Proposition 2. 1. *A right R-module B is F-flat if and only if its character module B^* is F-injective.*

Proposition 2. 2. *A right R-module B is R-flat if and only if, for every $I \in F(\mathfrak{T})$, the sequence $O \to B \otimes_R I \xrightarrow{1 \otimes i} B \otimes_R R$ is exact (where $I \xrightarrow{i} R$ is inclusion).*

(Note that the last condition is equivalent to say that $B \otimes_R I \cong BI$ canonically for every $I \in F(\mathfrak{T})$).

Proposition 2. 3. *Let $O \to K \to L \to B \to O$ is an exact sequence of right R-modules, where L is F-flat. Then B is F-flat if and only if $K \cap LI = KI$ for every $I \in F(\mathfrak{T})$.*

Each proof is omitted because it is essentially the same as was shown in Rotman [7], p 58—60.

Proposition 2. 4. *The following statements are equivalent for a right R-module B.*

(i) *B is F-flat.*

(ii) *Every exact sequence of right R-modules $O \to G \xrightarrow{\sigma} H \xrightarrow{\rho} B \to O$, tensoring by every $M \in \mathfrak{T}$, gives*

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an exactness of $O \longrightarrow G \otimes_R M \xrightarrow{\sigma\otimes 1} H \otimes_R M \xrightarrow{\rho\otimes 1} B \otimes_R M \longrightarrow O$.

(iii) *There exists an exact sequence of right R-modules* $O \longrightarrow G \xrightarrow{\sigma} H \xrightarrow{\rho} B \longrightarrow O$ *(where H is F-flat) which induces the exactness of* $O \longrightarrow G \otimes_R M \xrightarrow{\sigma\otimes 1} H \otimes_R M \xrightarrow{\rho\otimes 1} B \otimes_R M \longrightarrow O$ *for every cyclic* $M \in \mathfrak{T}$.

Proof. (cf. Bourbaki [3]) (i)$\Longrightarrow$(ii). Take an exact sequence $O \longrightarrow K \xrightarrow{i} F \xrightarrow{p} M \longrightarrow O$ where F is free. Then we have a commutative diagram

$$
\begin{array}{ccccccc}
 & & & & & & O \\
 & & & & & & \downarrow \\
 & & G\otimes_R K & \xrightarrow{\sigma\otimes 1} & H\otimes_R K & \xrightarrow{\rho\otimes 1} & B\otimes_R K \\
 & & {\scriptstyle 1\otimes i}\downarrow & & {\scriptstyle 1\otimes i}\downarrow & & {\scriptstyle 1\otimes i}\downarrow \\
O & \longrightarrow & G\otimes_R F & \xrightarrow{\rho\otimes 1} & H\otimes_R F & \xrightarrow{\sigma\otimes 1} & B\otimes_R F \\
 & & {\scriptstyle 1\otimes p}\downarrow & & {\scriptstyle 1\otimes p}\downarrow & & \\
 & & G\otimes_R M & \xrightarrow{\sigma\otimes 1} & H\otimes_R M & & \\
 & & \downarrow & & & & \\
 & & O & & & &
\end{array}
$$

in which all columns and the first two rows are exact. Diagram chasing shows that the map $\sigma\otimes 1_M$ is a monomorphism.

(ii) $\Longrightarrow$ (iii). We observe that any free H will do.

(iii) $\Longrightarrow$ (i). For every $I \in F(\mathfrak{T})$, the sequence $O \longrightarrow I \xrightarrow{i} R \xrightarrow{p} M \longrightarrow O$, where $M = R/I$, is exact. So we have a commutative diagram

$$
\begin{array}{ccccccc}
 & & & & & & O \\
 & & & & & & \downarrow \\
 & & G\otimes_R I & \xrightarrow{1\otimes i} & G\otimes_R R & \xrightarrow{1\otimes p} & G\otimes_R M \\
 & & {\scriptstyle \sigma\otimes 1}\downarrow & & {\scriptstyle \sigma\otimes 1}\downarrow & & {\scriptstyle \sigma\otimes 1}\downarrow \\
O & \longrightarrow & H\otimes_R I & \xrightarrow{1\otimes i} & H\otimes_R R & \xrightarrow{1\otimes p} & H\otimes_R M \\
 & & {\scriptstyle \rho\otimes 1}\downarrow & & {\scriptstyle \rho\otimes 1}\downarrow & & \\
 & & B\otimes_R I & \xrightarrow{1\otimes i} & B\otimes_R R & & \\
 & & \downarrow & & & & \\
 & & O & & & &
\end{array}
$$

in which all columns and the first two rows are exact. Diagram chasing shows that the map $1_B\otimes i$ is a monomorphism. By Proposition 2. 2, B is F-flat.

An exact sequence of right R-modules $O \longrightarrow G \xrightarrow{\sigma} H$ is called *F-pure* if, for every $M \in \mathfrak{T}$, the associated sequence $O \longrightarrow G \otimes_R M \xrightarrow{\sigma\otimes 1} H \otimes_R M$ is exact. One also says that $\sigma(G)$ is F-pure in H.

Corollary 2. 5. *Let H be a right R-module and G a submodule. If H/G is F-flat, then the sequence* $O \longrightarrow G \longrightarrow H$ *is F-pure. The converse holds if H is F-flat.*

A submodule K of a right R-module B is called *F-impure* in B, if $K \neq B$ and K contains no F-pure submodule of B other than O. An *F-flat cover* of a non-zero right R-module E is an exact sequence of right R-modules $O \longrightarrow K \longrightarrow B \longrightarrow E \longrightarrow O$ for which (i) B is F-flat and (ii) K is F-impure in B.

Proposition 2. 6. *The following statements are equivalent for an exact sequence of right R-modules* $O \longrightarrow K \xrightarrow{i} B \xrightarrow{u} E \longrightarrow O$ *where* $E \neq O$ *and B is F-flat.*

(i) *It is an F-flat cover of E.*

(ii) *If u has a factorization* $u = wv : B \xrightarrow{w} C \xrightarrow{v} E$ *where C is F-flat and w is an epimorphism, then w must be an isomorphism.*

Proof. This is a consequence of Corollary 2.5.

Theorem 2. 7. *Every non-zero right R-module has an F-flat cover.*

Proof. Take an exact sequence $O \longrightarrow K \longrightarrow B \longrightarrow E \longrightarrow O$ where E is free. By Zorn's lemma, we have a maximal submodule P of K such that P is F-pure in B. By Corollary 2. 5, B/P is F-flat. It suffices to prove that K/P is F-impure in B/P. Suppose K has a submodule H such that H contains P properly and H/P is F-pure in B/P. For any $M \in \mathfrak{T}$, we have a commutative diagram (where all of the maps are the obvious ones).

$$\begin{array}{ccccc} P \otimes_R M & \longrightarrow & H \otimes_R M & \longrightarrow & H/P \otimes_R M \\ \downarrow & & \downarrow & & \downarrow \\ P \otimes_R M & \longrightarrow & B \otimes_R M & \longrightarrow & B/P \otimes_R M \end{array}$$

Diagram chasing shows that $H \otimes_R M \longrightarrow B \otimes_R M$ is a monomorphism. Therefore H is F-pure in B. By the maximality of P, $P = H$, a contradiction.

3. F-projectives.

In this section all modules will mean left R-modules.

Lemma 3. 1. *A module P is F-projective if and only if every diagram*

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \swarrow & \downarrow & & \\ O & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & O \end{array}$$

where Q is F-injective, $Q' \in \mathfrak{F}$ *and the row is exact, can be completed to a commutative diagram.*

We shall omit the proof, which is similar to that of Ratman [7], p 76.

Theorem 3. 2. *The following statements are equivalent for a hereditary torsion theory* $(\mathfrak{T}, \mathfrak{F})$.

(i) *Every* $I \in F(\mathfrak{T})$ *is F-projective.*

(ii) *If* $O \longrightarrow Q' \longrightarrow Q \xrightarrow{\pi} Q'' \longrightarrow O$ *is an exact sequence of modules such that Q is F-injective and* $Q' \in \mathfrak{F}$, *then* Q'' *is F-injective.*

(iii) *If* $O \longrightarrow P' \xrightarrow{i} P \longrightarrow P'' \longrightarrow O$ *is an exact sequence of modules such that P is F-projective and* $P'' \in \mathfrak{T}$, *then* P' *is F-projective.*

Proof. (i)$\Longrightarrow$(ii). In order to prove that Q'' is F-injective, we are given a homomorphism $I \xrightarrow{f} Q''$ with $I \in F(\mathfrak{T})$. By hypothesis, I is F-projective, so there is a homomorphism $I \xrightarrow{g} Q$ such that $\pi g = f$. Since Q is F-injective, g can be extended to a homomorphism $R \xrightarrow{h} Q$. Then πh has the desired property.

(ii)$\Longrightarrow$(iii). In order to prove P' is F-projective using Lemma 3. 1, we are given an exact sequence of modules $O \longrightarrow Q' \longrightarrow Q \xrightarrow{\pi} Q'' \longrightarrow O$, where Q is F-injective and $Q' \in \mathfrak{F}$, and a homomorphism $P' \xrightarrow{f} Q''$. By hypothesis, Q'' is F-injective, so f can be extended to a homomorphism $P \xrightarrow{g} Q''$. Since P is F-projective, there is a homomorphism $P \xrightarrow{h} Q$ such that $\pi h = g$. Then hi has the desired property.

(iii)$\Longrightarrow$(i). Trivial.

An *F-projective cover* of a module A is an exact sequence of modules $O \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow O$ for which (i) P is F-projective, (ii) $K \in \mathfrak{F}$ and (iii) K is small in P.

Proposition 3. 3 (*Uniqueness*). *For any two F-projective covers* $O \longrightarrow K_i \longrightarrow P_i \xrightarrow{\pi_i} A \longrightarrow O$ $(i = 1, 2)$ *of A, there exists an isomorphism* $P_1 \xrightarrow{\theta} P_2$ *satisfying* $\pi_2 \theta = \pi_1$.

Proof. Since P_1 is F-projective, there exists a homomorphism $P_1 \xrightarrow{\theta} P_2$ satisfying $\pi_2 \theta = \pi_1$.

Then we have $P_2 = \mathrm{Im}\ \theta + \mathrm{Ker}\ \pi_2$. Since K_2 is small in P_2, $P_2 = \mathrm{Im}\ \theta$. Thus θ is an epimorphism. Now $\mathrm{Ker}\ \theta \subset \mathrm{Ker}\ \pi_1$, so $\mathrm{Ker}\ \theta \in \mathfrak{F}$ and $\mathrm{Ker}\ \theta$ is small in P_1. Since P_2 is F-projective, there exists a homomorphism $P_2 \xrightarrow{\psi} P_1$ with $\theta\psi = 1_{P_2}$. Thus $P_1 = \mathrm{Im}\ \psi \oplus \mathrm{Ker}\ \theta$. So, by $\mathrm{Ker}\ \theta = O$, θ is a monomorphism.

Proposition 3. 4 (*cf*, [6] *Proposition* 3. 3). *Let* $O \longrightarrow K \longrightarrow P \xrightarrow{e} A \longrightarrow O$ *be an exact sequence of modules with* $K \in \mathfrak{T}$. *If P is F-projective, then so is A.*

Proof. Let an exact sequence $O \longrightarrow M' \longrightarrow M \xrightarrow{\delta} M'' \longrightarrow O$ with $M' \in \mathfrak{F}$ and a homomorphism $A \xrightarrow{f} M''$ be given. Since P is F-projective, there exists a homomorphism $P \xrightarrow{g} M$ making the square

$$\begin{array}{ccccccccc} & & & & P & \xrightarrow{e} & A & & \\ & & & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \\ O & \longrightarrow & M' & \longrightarrow & M & \xrightarrow{\delta} & M'' & \longrightarrow & O \end{array}$$

commutative. Thus $g(K) \subset M'$, so $g(K) \in \mathfrak{F}$. On the other hand, since $K \in \mathfrak{T}$, $g(K) \in \mathfrak{T}$. Therefore $g(K) = O$. Hence g induces a homomorphism $A \xrightarrow{h} M$, and we can easily show that $\delta h = f$.

Theorem 3. 5 (*Existence*). *Let R be left perfect. Then every* $A \in \mathfrak{F}$ *has an F-projective cover.*

Proof. Let $O \longrightarrow K \longrightarrow P \xrightarrow{\pi} A \longrightarrow O$ be a projective cover of A. Since $A_t = O$, $\pi(P_t) = O$. Thus π induces the exact sequence $O \longrightarrow (K + P_t)/P_t \longrightarrow P/P_t \longrightarrow A \longrightarrow O$. By Proposition 3. 4, P/P_t is F-projective. Since $P/P_t \in \mathfrak{F}$, $(K + P_t)/P_t \in \mathfrak{F}$. By the condition that K is small in P, we observe that $(K + P_t)/P_t$ is small in P/P_t.

Example. Let R be a quasi-Frobenius ring but not a semi-simple Artinian, and let $(\mathfrak{G}, \mathfrak{R})$ be Goldie's torsion theory for ${}_R\mathfrak{M}$. Then, as was shown by Armendariz [1], every module in $\mathfrak{R}$ is injective. Hence every module is F-projective, but there is a module which is not projective.

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