AN ALMOST DERIVED ANALOGUE OF NAKAYAMA'S LEMMA

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Abstract. As Nakayama's lemma, in commutative ring theory, it is well-known that the residue functor of an ideal contained in the Jacobson radical is conservative. In almost mathematics setting, considering an almost version of perfect complexes, we prove an almost derived analogue of Nakayama's lemma, stating the conservativity of the derived functor of the residue of a tight ideal contained in the Jacobson radical.

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1. Introduction

Nakayama's lemma is one of the most crucial theorem in commutative ring theory:

Theorem 1.1 (Nakayama's lemma)**.** *Let V be a commutative ring and I an ideal of V contained the Jacobson radical of V, and let $\mathrm{Mod}^{\mathrm{f}}_V$ denote the category of finitely generated V-modules. Then the functor*

$$(-) \otimes_V (V/I) : \mathrm{Mod}^{\mathrm{f}}_V \to \mathrm{Mod}^{\mathrm{f}}_{V/I}$$

is conservative. That is, M/IM is trivial if and only if so is M. □

In almost mathematics setting, Gabber and Ramero [1] proved an analogue of Nakayama's lemma [1, Lemma 5.1.7].

We explain a brief introduction of almost mathematics. In this paper, we fix a unital commutative ring V. Let $\mathfrak{m}$ be an idempotent ideal of V. We always assume that $\tilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module.

A V-module M is said to be *almost zero* if $\mathfrak{m}M = 0$. Then the full subcategory of almost zero modules of the category of V-modules is Serre subcategory, and let alMod_V denote the localization of the category of V-modules by the Serre subcategory of almost zero modules.

A V-module M is said to be *firm* if the product $\mu_M : \mathfrak{m} \otimes_V M \to M$ is an isomorphism. It is well-known that, for any V-module M, $\tilde{\mathfrak{m}} \otimes_V M$ is always firm. Therefore, let $\mathrm{Mod}^{\mathrm{fi}}_V$ denote the full subcategory of Mod_V spanned by firm V-modules. Quillen [3] proved that the functor

$$\tilde{\mathfrak{m}} \otimes_V (-) : \mathrm{Mod}_V \to \mathrm{Mod}^{\mathrm{fi}}_V$$

induces an categorical equivalence from alMod_V to $\mathrm{Mod}^{\mathrm{fi}}_V$. Therefore, a homomorphism of $f : M \to N$ is an isomorphism in the category alMod_V if and only if the induce map $\tilde{\mathfrak{m}} \otimes_V f : \tilde{\mathfrak{m}} \otimes_V M \to \tilde{\mathfrak{m}} \otimes_V N$ is an isomorphism of V-modules. We say that a homomorphism $f : M \to N$ of V-modules is *almost injective* (resp. *almost surjective*, *an almost isomorphism*) if the induced morphism $\tilde{\mathfrak{m}} \otimes_V f : \tilde{\mathfrak{m}} \otimes_V M \to \tilde{\mathfrak{m}} \otimes_V N$ is injective (resp. surjective, an isomorphism).

Dually, if the induced map $\mu'_M : M \to \mathrm{Hom}_V(\mathfrak{m}, M)$ by $\mu_M : \mathfrak{m} \otimes_V M \to M$ is an isomorphism, we say that M is *closed*. it is easily checked that, for any V-module M, the V-module $M_* := \mathrm{Hom}_V(\tilde{\mathfrak{m}}, M)$ is closed.

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We use the following lemma in this paper:

Lemma 1.2. *For any V-module M, the induced homomorphisms $\tilde{\mathfrak{m}} \otimes_V M \to M$ and $M \to M_*$ are almost isomorphisms.*

proof. See [3, Section 2 and Section 5]. □

A commutative monoid object of the abelian category alMod_V is called an *almost V-algebra*.

Definition 1.3. Let A be an almost V-algebra. An A-module M is *almost finitely generated* (resp. *almost finitely presented*) if for any filtered inductive system (N_α) of firm A-module, the canonical map

$$\varinjlim \mathrm{Hom}_{\mathrm{Mod}_A^{\mathrm{fi}}}(\tilde{\mathfrak{m}} \otimes_V M, N_\alpha) \to \mathrm{Hom}_{\mathrm{Mod}_A^{\mathrm{fi}}}(\tilde{\mathfrak{m}} \otimes_V M, \varinjlim N_\alpha)$$

is almost injective (resp. an almost isomorphism).

Definition 1.4. Let A be an almost V-algebra. The *Jacobson radical* is defined to be $\mathrm{Jac}(A) = \mathrm{Jac}(A_*)^a \subset A$, where $(-)^a : \mathrm{Mod}_V \to \mathrm{alMod}_V$ denotes the localization functor.

Following Gabber and Ramero [1, section 5], we introduce tight ideals to formulate an almost version of Nakayama's lemma:

Definition 1.5. Let A be an almost V-algebra and I an ideal of A. We say that I is *tight* if there exists a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and an integer $n \geq 0$ such that $I^n \subset \mathfrak{m}_0 A$.

The almost version of Nakayama's lemma is the following:

Theorem 1.6 ([1] Lemma 5.1.7.)**.** *Let A be an almost V-algebra, I a tight ideal of A which is contained in the Jacobson radical of A. If M is an almost finitely generated A-module satisfying $IM = M$, then exactly $M = 0$.* □

Corollary 1.7 ([1] Corollary 5.1.8.)**.** *Let A be an almost V-algebra, I a tight ideal of A which is contained in the Jacobson radical of A. Let $f : M \to N$ be a homomorphism of almost finitely generated projective A-modules. If $f \otimes_A A/I : M/IM \to N/IN$ is an isomorphism. Then f is also an isomorphism.* □

Introducing an almost version of perfect complexes by the following [2], this note provides an almost derived version of Theorem 1.6.

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2. Almost perfect complexes

We explain the definition of almost perfect complexes, being equivalent one in [2]:

Definition 2.1. An A-module P is *almost projective* if the functor $\mathrm{Hom}_{\mathrm{alMod}_A}(P, -) : \mathrm{alMod}_A \to \mathrm{alMod}_A$ is exact, equivalently, $\mathrm{Hom}_{\mathrm{Mod}_A^{\mathrm{fi}}}(\tilde{\mathfrak{m}} \otimes_V P, -) : \mathrm{Mod}_A^{\mathrm{fi}} \to \mathrm{alMod}_A$ is exact.

Any projective A-module is also almost projective. Let alPMod_A denote the full subcategory of Mod_A spanned by those objects $\tilde{\mathfrak{m}} \otimes_V P$ where P is a finitely generated projective A-module.

Proposition 2.2. *An object of* alPMod_A *is almost finitely presented projective.*

proof. Since P is a direct factor of finitely generated free A-module, it is sufficient to check that $\tilde{\mathfrak{m}} \otimes_V A$ is an almost finitely generated projective A-module. Note that one has a canonical isomorphism $\mathrm{Hom}_{\mathrm{Mod}_A^{\mathrm{fi}}}(\tilde{\mathfrak{m}} \otimes_V A, N) \simeq \mathrm{Hom}_{\mathrm{Mod}_V^{\mathrm{fi}}}(\tilde{\mathfrak{m}}, N) \simeq N_*$ for any A-module N. Therefore, by Lemma 1.2, one has a chain of isomorphisms:

$$\begin{aligned}\tilde{\mathfrak{m}} \otimes_V \varinjlim \mathrm{Hom}_{\mathrm{Mod}_A^{\mathrm{fi}}}(\tilde{\mathfrak{m}} \otimes_V A, N_\alpha) &\simeq \tilde{\mathfrak{m}} \otimes_V \varinjlim (N_\alpha)_* \\ &\simeq \varinjlim \tilde{\mathfrak{m}} \otimes_V (N_\alpha)_* \simeq \varinjlim \tilde{\mathfrak{m}} \otimes_V N_\alpha \simeq \tilde{\mathfrak{m}} \otimes_V \varinjlim N_\alpha \simeq \tilde{\mathfrak{m}} \otimes_V \mathrm{Hom}_{\mathrm{Mod}_A^{\mathrm{fi}}}(\tilde{\mathfrak{m}} \otimes_V A, \varinjlim N_\alpha),\end{aligned}$$

implying that $\tilde{\mathfrak{m}} \otimes_V A$ is almost finitely presented. For any exact sequence $M' \to M \to M''$ of firm A-modules, the induced sequence $M'_* \to M_* \to M''_*$ is exact after tensoring with $\tilde{\mathfrak{m}}$ by Lemma 1.2, again. □

Definition 2.3. Let $\mathrm{APerf}(A)$ denote the triangulated full subcategory of $D(\mathrm{Mod}_A^{\mathrm{fi}})$ spanned by complexes which are quasi-isomorphic to some bounded complex of alPMod_A.

By Proposition 2.2, an object of $\mathrm{APerf}(A)$ is represented by some bounded complex of almost finitely generated projective A-modules.

The main result is the following:

Theorem 2.4. *Let A be a V-algebra and I a tight ideal which is contained in the Jacobson radical of A. Let E be an almost perfect A-complex. Then the functor $- \otimes_A^{\mathbb{L}} A/I : \mathrm{APerf}(A) \to \mathrm{APerf}(A/I)$ is conservative.*

proof. We will prove the theorem by induction of the length of almost perfect complexes. In the case of the length 0, the assertion is trivial. Let $f : E_1 \to E_0$ be a morphism of almost finitely generated projective A-modules. If $\mathrm{Coker} f \otimes_A A/I = 0$, then $\mathrm{Coker} f = 0$ by Corollary 1.7.

Let $E = (E_i)_{0 \le i \le n}$ be a complex of almost finitely generated projective A-modules of length n. Assume that $E \otimes_A A/I$ is almost acyclic. Consider the distinguished triangle:

$$\tau_{\le -2} E \to E \to \tau_{\ge -1} E \to \tau_{\le 1} E[1],$$

where $\tau_{\ge -1} E : 0 \to B_0 \to E_0 \to 0$ and $\tau_{\le -2} E : 0 \to E_n \to E_{n-1} \to \cdots \to E_2 \to Z_1 \to 0$. Since $\tilde{\mathfrak{m}}$ is V-flat, the functor $\tilde{\mathfrak{m}} \otimes_V (-)$ is exact. Therefore it is commutative with truncation functors. Note that any truncation functor preserving quasi-isomorphisms by definition. Therefore $\tau_{\le -2} E \otimes_A A/I$ and $\tau_{\ge -1} E \otimes_A A/I$ are almost acyclic. By definition of B_0 and exactness of $\tilde{\mathfrak{m}} \otimes_V (-)$, the map $B_0 \to E_0$ is almost injective. The condition that $B_0 \otimes_A A/I \to E_0 \otimes_A A/I$ is almost surjective implies that $B_0 \to E_0$ is surjective by Theorem 1.6. In particular B_0 is almost finitely generated projective over A, implying that Z_1 is also almost finitely generated projective. By the assumption of induction, $\tau_{\le -2} E$ is almost acyclic. We obtain that E is also almost acyclic. □

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