BILOCALIZATION OF ABELIAN CATEGORIES

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ABSTRACT. This note is a revisiting Quillen's note [3] "Modules theory over non-unital rings" in category theory language. Faltings [1]; Gabber and Ramero [2] established almost mathematics which is the same as Quillen's bilocalization of a category of modules by almost zero objects.

Keywords: Bilocalization, non-unital algebras, almost mathematics.

1. INTRODUCTION

Faltings first introduced almost mathematics in his article [1], proving almost purity. After Gabber and Ramero established almost ring theory: almost modules, almost algebras, and almost homotopical algebras in their text book [2]. While almost mathematics has various applications to arithmetic geometry, for example, perfectoid geometry, Quillen [3] mentioned linear algebra over non-unital rings which is the same as almost mathematics. Quillen's work is conceptional: in his work, almost mathematics is characterized as bilocalization of an abelian category of modules.

Theorem 1.1 ([3] Proposition 6.5)**.** *There is a one-to-one correspondence between Serre subcategories $\mathcal{S}$ of Mod_V which the localization $F : \mathrm{Mod}_V \to \mathrm{Mod}_V/\mathcal{S}$ is also a colocalization, and full subcategories spanned by almost I-zero modules for some idempotent ideal I of V.* □

This note provides a more conceptional definition of almost mathematics in category language to extend the theory to more general settings, like ∞-categories, by using formal category theory arguments as far as possible.

In Section 2, we generalize Quillen's work [3] to abelian categories, and prove the existence of the canonical categorical equivalences between the full subcategory spanned by local objects and the one spanned by colocal objects.

In Section 3, specializing results of Section 2, we redefine almost modules in almost mathematics by using Gabber and Ramero's [2] terminology. Finally, we give a proof of Theorem 1.1 by using the elemental limit argument in order to generalize the theorem for ∞-categories.

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2. BILOCALIZATION OF CATEGORIES

This section provides a brief review of bilocalizations of abelian categories. All results of this section are obtained by formal arguments of category theory.

Definition 2.1. Let $F : \mathcal{C} \to \mathcal{M}$ be a functor of categories.

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(1) The functor F is a localization of $\mathcal{C}$ if F admits a fully faithful right adjoint $F_! : \mathcal{M} \to \mathcal{C}$.
(2) The functor F is a colocalization of $\mathcal{C}$ if F admits a fully faithfully left adjoint $F_* : \mathcal{M} \to \mathcal{C}$.
(3) The functor F is a bilocalization of $\mathcal{C}$ if F is both a localization and colocalization.

Note that $F : \mathcal{C} \to \mathcal{M}$ is a localization if and only if the opposite $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$ is a colocalization, where $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{M}^{\mathrm{op}}$ are the opposite categories.

The following is a generalization of almost zero modules and almost isomorphisms of almost mathematics.

Definition 2.2. Let $F : \mathcal{C} \to \mathcal{M}$ be a bilocalization of an abelian category $\mathcal{C}$. An object M is said to be F-nil if $F(M) = 0$, and a morphism is an F*-isomorphism* if its kernel and cokernel are F-nil objects.

Proposition 2.3. *Let $F : \mathcal{C} \to \mathcal{M}$ be a bilocalization. For any object M of $\mathcal{C}$, the counit morphism $\mu_M : F_*(F(M)) \to M$ of the adjunction (F_*, F) and the unit morphism $\mu'_M : M \to F_!(F(M))$ of the adjunction $(F, F_!)$ are both F-isomorphisms.*

proof. The proposition follows from fully faithfulness of F_* and $F_!$. We only prove that the induced morphism $F(\mu) : F(F_*(F(M))) \to F(M)$ is an isomorphism. The second assertion is dual. For any object N, indeed, the induced map $F(\mu)_* : \mathrm{Hom}_{\mathcal{M}}(N, F(M)) \to \mathrm{Hom}_{\mathcal{M}}(N, F(F_*(F(M)))))$ factors as a chain of isomorphisms:

$$\mathrm{Hom}_{\mathcal{M}}(N, F(M)) \simeq \mathrm{Hom}_{\mathcal{C}}(F_*(N), F_*(F(M))) \simeq \mathrm{Hom}_{\mathcal{M}}(N, F(F_*(F(M)))).$$

By Yoneda lemma, $F(\mu)$ is an isomorphism. □

Definition 2.4. Let $F : \mathcal{C} \to \mathcal{M}$ be a localization. An object M of $\mathcal{C}$ is F*-local* if any F-isomorphism $f : N_1 \to N_2$, the induced map $f_* : \mathrm{Hom}_{\mathcal{C}}(M, N_1) \to \mathrm{Hom}_{\mathcal{C}}(M, N_2)$ is bijective. Similarly, let $F : \mathcal{C} \to \mathcal{M}$ be a colocalization. An object M of $\mathcal{C}$ is F*-colocal* if any F-isomorphism $f : N_1 \to N_2$, the induced map $f^* : \mathrm{Hom}_{\mathcal{C}}(N_2, M) \to \mathrm{Hom}_{\mathcal{C}}(N_1, M)$ is bijective.

Proposition 2.5. *Let $F : \mathcal{C} \to \mathcal{M}$ be a localization of abelian category $\mathcal{C}$ and M an object of $\mathcal{C}$. Then the following conditions are equivalent:*
(1) The object M is F-local.
(2) The counit map $\mu : F_*(F(M)) \to M$ is already an isomorphism.
(3) For $i = 0, 1$ and any F-nil object N, $\mathrm{Ext}^i_{\mathcal{C}}(M, N) = 0$.

proof. We show the implication (2) to (3). Assume that $\mu : F_*(F(M)) \to M$ is an isomorphism. Then μ induces a chain of isomorphisms $\mathrm{Hom}_{\mathcal{C}}(M, N) \simeq \mathrm{Hom}_{\mathcal{M}}(F(M), F(N)) \simeq \mathrm{Hom}_{\mathcal{M}}(F(M), 0) = 0$ for any F-nil object N. Given $x \in \mathrm{Ext}^1_{\mathcal{C}}(M\, N)$ corresponding to an extension $0 \to N \to E \xrightarrow{x} M \to 0$, one has a long exact sequence

$$0 \to \mathrm{Hom}(M, E) \to \mathrm{Hom}(M, M) \to \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M, E) \to \cdots .$$

Since F is an exact functor and N is F-nil, the induced map $F(x) : F(E) \to F(M)$ is an isomorphism. Therefore one has a chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}(M, E) \simeq \mathrm{Hom}(F_*(F(M)), E) \simeq\ & \mathrm{Hom}(F(M), F(E)) \\ & \simeq \mathrm{Hom}(F(M), F(M)) \simeq \mathrm{Hom}(F_*(F(M)), F(M)) \simeq \mathrm{Hom}(M, M).\end{aligned}$$

Hence x has a section, implying $x = 0$ in $\mathrm{Ext}^1_{\mathcal{C}}(M, N)$.

Clearly, the condition (1) follows from (3): Let $f : N_1 \to N_2$ be an F-isomorphism. Short exact sequences $0 \to \mathrm{Im}f \to N_2 \to \mathrm{Coker}f \to 0$ and $0 \to \mathrm{Ker}f \to N_1 \to \mathrm{Im}f \to 0$ induce isomorphisms $\mathrm{Hom}(M, \mathrm{Im}f) \to \mathrm{Hom}(M, N_2)$ and $\mathrm{Hom}(M, N_1) \to \mathrm{Hom}(M, \mathrm{Im}f)$.

Assume that the condition (1) holds. Then, by Proposition 2.3, one has an isomorphism $\mathrm{Hom}(M, M) \simeq \mathrm{Hom}(M, F_*(F(M)))$, implying that $\mu : F_*(F(M)) \to M$ has a section. Similarly, the injectivity of μ is obtained by isomorphisms $\mathrm{Hom}(F_*(F(M)), F_*(F(M))) \simeq \mathrm{Hom}(F(M), F(M)) \simeq \mathrm{Hom}(F_*(F(M)), M)$. □

Considering opposite categories in Proposition 2.5, we have the following:

Proposition 2.6. *Let $F : \mathcal{C} \to \mathcal{M}$ be a colocalization. The following conditions are equivalent:*

(1) An object M is F-colocal.
(2) The unit $\mu' : M \to F_! F(M)$ is already an isomorphism.
(3) For $i = 0, 1$ and any F-nil object N, $\mathrm{Ext}^i_{\mathcal{C}}(N, M) = 0$. □

The main result of this section is the following:

Theorem 2.7. *Let $F : \mathcal{C} \to \mathcal{M}$ be a bilocalization of an abelian category $\mathcal{C}$. Let $\mathcal{C}_{F\text{-}loc}$ denote the full subcategory of $\mathcal{C}$ spanned by all F-local objects, and $\mathcal{C}_{F\text{-}col}$ the full subcategory of $\mathcal{C}$ spanned by all F-colocal objects. Then the induced adjunctions*

$$\mathcal{C}_{F\text{-}col} \underset{F_!}{\overset{F}{\rightleftarrows}} \mathcal{M} \underset{F}{\overset{F_*}{\rightleftarrows}} \mathcal{C}_{F\text{-}loc}$$

are categorical equivalences.

proof. We only prove that the left two functors are categorical equivalences. The proof of other equivalences is similar. By Proposition 2.6, $F_! \circ F \simeq \mathrm{Id}_{\mathcal{C}_{F\text{-col}}}$. Since $F_!$ is fully faithful, one has a chain of isomorphisms:

$$\mathrm{Hom}_{\mathcal{M}}(F(F_!(M)), N)) \simeq \mathrm{Hom}_{C_{F\text{-col}}}(F_!(M), F_!(N)) \simeq \mathrm{Hom}_{\mathcal{M}}(M, N)$$

for any objects M and N of $\mathcal{M}$. By Yoneda Lemma, $F(F_!(M)) \simeq M$. Hence $F \circ F_! \simeq \mathrm{Id}_{\mathcal{M}}$. □

3. Comparison to almost mathematics

In this section we consider bilocalizations of a category of modules over a ring V, which is not necessarily commutative.

Let I be a two-sided idempotent ideal of V. A V-module M is said to be *almost I-zero (or simply, almost zero)* if $IM = 0$. A V-homomorphism $f : M \to N$ of V-modules is called *an almost I-isomorphism* if both the kernel and the cokernel of f are almost I-zero.

Quillen [3] proved that giving a bilocalization $F : \mathrm{Mod}_V \to \mathcal{M}$ corresponds to determining an idempotent ideal I of V such that $\mathcal{M}$ is a bilocalization by the Serre subcategory of almost I-zero modules.

Lemma 3.1. *Let V be a unital ring and I an idempotent ideal. Then $I \otimes_V M = 0$ if and only if M is almost I-zero.*

proof. The only if direction is clear. Assume $IM = 0$. For $a \otimes x \in I \otimes_V M$ ($a \in I$, $x \in M$), there exists $(a_i b_i) \subset I^2$ such that $a = \sum_i a_i b_i$. Hence one has $a \otimes x = (\sum_i a_i b_i) \otimes x = \sum_i a_i b_i \otimes x = \sum_i a_i \otimes b_i x = 0$. □

Proposition 3.2. *Let V be a unital ring, I an idempotent ideal of V, and M a V-module. Then the canonical morphisms $\mu : I \otimes_V M \to M$ and $\mu' : M \to \mathrm{Hom}_V(I, M)$ are both almost isomorphisms.*

proof. Since the abelian category Mod_V is both enough projective and injective, the kernels and the cokernels; $\mathrm{Tor}_i^V(V/I, M)$ and $\mathrm{Ext}_V^i(V/I, M)$ for $i = 0, 1$, are killed by I. □

Definition 3.3. In the situation of Proposition 3.2, we say that M is *I-firm* if μ is an isomorphism and M is *I-closed* if μ' is an isomorphism.

Corollary 3.4. *Let I be an idempotent ideal of a unital ring V. Then the map $\mu_I : I \otimes_V I \to I$ is an almost I-isomorphism and $\mu_I \otimes I : I \otimes_V I \otimes_V I \to I \otimes_V I$ already an isomorphism.*

proof. This follows from Lemma 3.1 and Proposition 3.2. □

Corollary 3.5 (c.f.[3] Proposition 4.1 and Proposition 5.3)**.** *Let V be a unital ring and I an idempotent ideal of V. Write $\tilde{I} = I \otimes_V I$. For any V-module M, $\tilde{I} \otimes_V M$ is I-firm and* $\mathrm{Hom}(\tilde{I}, M)$ *I-closed.*

proof. By Corollary 3.4, the counit $\mu_{\tilde{I}\otimes_V M} : I \otimes_V \tilde{I} \otimes_V M \to \tilde{I} \otimes_V M$ is an isomorphism, inducing canonical isomorphisms $\mathrm{Hom}(\tilde{I}, M) \simeq \mathrm{Hom}(I \otimes_V \tilde{I}, M) \simeq \mathrm{Hom}(I, \mathrm{Hom}(\tilde{I}, M))$. □

Let $F : \mathrm{Mod}_V \to \mathrm{Mod}_V/\mathcal{S}$ be a bilocalization. An $\mathcal{S}$-local (resp. $\mathcal{S}$-colocal) object is an F-local (resp. F-colocal) object.

Lemma 3.6 (c.f.[3] Proposition 4.1 and Proposition 5.3)**.** *Let V be a unital ring, I an idempotent ideal of V and M a V-module. Let $\mathcal{S}$ denote the Serre subcategory of almost I-zero modules of* Mod_V*. Then the following conditions are equivalent:*

(1) The V-module M is $\mathcal{S}$-local.

(2) The V-module M is I-firm.

Similarly, the following conditions are equivalent:

(1)' The V-module M is $\mathcal{S}$-colocal.

(2)' The V-module M is I-closed.

proof. Assume that M is $\mathcal{S}$-local. By Proposition 3.2, the induced map

$$\mu_* : \mathrm{Hom}(M, I \otimes_V M) \to \mathrm{Hom}(M, M)$$

is bijective. Hence μ is surjective, in particular $M = IM$. Since M is a direct factor of $I \otimes_V M$, the composition $M \to I \otimes_V M \to I \otimes_V I \otimes_V M$ is split. A direct factor M of the I-firm module $\tilde{I} \otimes_V M$ is also I-firm.

Assume that M is an I-firm module. Let N be an almost I-zero module. We prove that $\mathrm{Ext}_V^i(M, N) = 0$ for $i = 0, 1$. The condition $IN = 0$ implies that N is also V/I-module. The adjunction between the derived categories:

$$(V/I) \otimes_V^{\mathbb{L}} - : D(V) \rightleftarrows D(V/I) : \mathbb{R}\mathrm{Hom}_V(V/I, -)$$

induces a canonical equivalence $\mathbb{R}\mathrm{Hom}_V((V/I) \otimes_V^{\mathbb{L}} M, N) \simeq \mathbb{R}\mathrm{Hom}_V(M, N)$ and the spectral sequence

$$E_2^{pq} = \mathrm{Ext}_{V/I}^p(\mathrm{Tor}_q^V(V/I, M), N) \Longrightarrow \mathrm{Ext}_V^{p+q}(M, N)$$

where $E_2^{pq} = 0$ for $q = 0, 1$. Hence one has $\mathrm{Ext}_V^0(M, N) = \mathrm{Ext}_V^1(M, N) = 0$, implying that M is $\mathcal{S}$-local by Proposition 2.5.

Assume that M is $\mathcal{S}$-colocal. Consider the induced exact sequence:

$$0 \to \mathrm{Hom}_V(V/I, M) \to M \to \mathrm{Hom}_V(I, M) \to \mathrm{Ext}_V^1(V/I, M) \to 0.$$

By the assumption and Proposition 2.6, one has $\mathrm{Hom}_V(V/I, M) = \mathrm{Ext}_V^1(V/I, M) = 0$.

Finally, if M is closed, then one has $\mathrm{Hom}_V(N, \mathrm{Hom}_V(\tilde{I}, M)) \simeq \mathrm{Hom}_V(N \otimes_V \tilde{I}, M) = 0$ for any almost I-zero module N. □

Theorem 3.7 (c.f.[3] Theorem 4.5 and Theorem 5.6)**.** *Let V be a unital ring and I an idempotent ideal of V. Let $\mathcal{S}$ denote the Serre subcategory of almost I-zero modules of* Mod_V*. Then the localization functor $F : \mathrm{Mod}_V \to \mathrm{Mod}_V/\mathcal{S}$ is also a colocalization. Furthermore, the left adjoint is isomorphic to the functor $\tilde{I} \otimes_V - : \mathrm{Mod}_V/\mathcal{S} \to \mathrm{Mod}_V$ whose essential image is the full subcategory of I-firm modules, and the right adjoint is isomorphic to* $\mathrm{Hom}_V(\tilde{I}, -) : \mathrm{Mod}_V/\mathcal{S} \to \mathrm{Mod}_V$ *whose essential image is the full subcategory of I-closed modules.* □

Conversely, let $F : \mathrm{Mod}_V \to \mathrm{Mod}_V/\mathcal{S}$ be a bilocalization and $\mathcal{S}$ denotes the full subcategory of Mod_V spanned by F-nil objects. Since F preserves all small limits and colimits, $\mathcal{S}$ is closed under small limits and colimits. In particular, it is closed under products and direct sums. Since a single object V generates the presentable abelian category Mod_V, any module M forms an iterated colimits of $V/\ell V$ for some $\ell \in V$. Set

$$I = \bigcap_{\ell_\alpha \in V : F(V/\ell_\alpha V) = 0} \ell_\alpha V. \tag{3.1}$$

Then I is a two-sided ideal of V.

We show that $F(M) = 0$ if and only if M is killed by I.

Lemma 3.8. *Let $F : \mathrm{Mod}_V \to \mathrm{Mod}_V/\mathcal{S}$ be a bilocalization. For any $\ell \in V$, $F(V/\ell V) = 0$ if and only if the induced map $F(\ell) : F(V) \to F(V)$ is bijective.*

proof. Note that F is an exact functor. The if direction is clear. The condition $F(V/\ell V) = 0$ implies that $F(\ell V) = F(V)$ and $F(\ell)$ is surjective. Applying F to a short exact sequence $0 \to \mathrm{Ann}(\ell) \to V \to \ell V \to 0$, we obtain $F(\mathrm{Ann}(\ell)) = 0$. Hence $F(\ell)$ is injective. □

Lemma 3.9. *Let $F : \mathrm{Mod}_V \to \mathrm{Mod}_V/\mathcal{S}$ be a bilocalization. Consider an ideal*

$$I = \bigcap_{\ell_\alpha \in V : F(V/\ell_\alpha V) = 0} \ell_\alpha V.$$

Then the canonical homomorphism

$$\varphi : I \to \varprojlim_{\ell_\alpha \in V : F(\ell_\alpha) \text{ is isomorphic.}} (\cdots \to V \xrightarrow{\ell_\alpha} V)$$

is an isomorphism.

proof. Since the map $\ell : V \to V$ factors as $\ell : V \to \ell V \to V$, the above two limits are isomorphic by Lemma 3.8. □

Proposition 3.10 (c.f.[3] 6.4)**.** *For any V-module, $F(M) = 0$ if and only if $IM = 0$ where I is the ideal in 3.1.*

proof. It is sufficient to prove in the case $M = V/\ell V$. The bijectivity of $F(\ell)$ implies that $\ell : I \to I$ is an isomorphism by Lemma 3.9. Hence $I/\ell I = I \otimes_V (V/\ell V) = 0$ and $I \cdot (V/\ell V) = 0$.

Conversely, we prove that $I/\ell I = 0$ implies $F(\ell I) = F(I)$. Since F preserves all small limits, one has

$$F(I) = \bigcap_{\ell_\alpha \in V : F(V/\ell_\alpha V) = 0} F(\ell_\alpha V) = \bigcap_{\ell_\alpha \in V : F(V/\ell_\alpha V) = 0} F(V) = F(V).$$

Hence $F(V) = F(I) = F(\ell I)$. Note that a bilocalization is an exact functor. The inclusion : $\ell I \to \ell V$ induces an inclusion $F(\ell I) \to F(\ell V)$. We have $F(V) = F(\ell V)$ and $F(V/\ell V) = 0$. □

Proposition 3.11 ([3] 6.4)**.** *The ideal I in 3.1 is idempotent.*

proof. Consider the exact sequence:

$$0 \to I/I^2 \to V/I^2 \to V/I \to 0.$$

The modules I/I^2 and V/I are killed by I. Therefore $F(I/I^2) = F(V/I) = 0$. The exactness of F gives $F(V/I^2) = 0$, by Proposition 3.10, $I \cdot (V/I^2) = 0$. That is, for any $\ell \in I$, $\ell \cdot 1 \in I^2$. Thus $I \subset I^2$. □

Theorem 3.12 ([3] 6.4)**.** *Let $F : \mathrm{Mod}_V \to \mathrm{Mod}_V/\mathcal{S}$ be a bilocalization by a Serre subcategory $\mathcal{S}$ of Mod_V. Then there exists an idempotent ideal of I such that $\mathcal{S}$ coincides with the full subcategory spanned almost I-zero modules.* □

From Theorem 3.7 and Theorem 3.12, Theorem 1.1 is obtained.

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