Double inequalities derived from the arithmetic-geometric-harmonic mean inequalities with power exponential functions

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Abstract: In this paper, we give double inequalities derived from the arithmetic-geometric-harmonic mean inequalities with power exponential functions.

Keywords: mean inequality, monotonically increasing function, monotonically decreasing function.

1. Introduction

Studies of inequalities are active areas in the mathematical analysis. Arithmetic-geometric-harmonic mean inequality of two variable numbers is known as that; for $a > 0$ and $b > 0$, the inequality

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \tag{0.1}$$

holds. Recently, Chen et al. [1, 2, 3, 4, 5] gave the double inequalities with power exponential functions in which the base of the left hand side is equal to the base of the right hand side. In this paper, we give double inequalities which have the above feature. In Theorems 1 and 2, we establish the reversed Arithmetic-geometric-harmonic mean inequality of two variable numbers with power exponential functions as follows.

Theorem 1. *For $0 < b \leq a \leq 1$, we have*

$$\left(\frac{a+b}{2}\right)^{\theta(b)} \leq \sqrt{ab} \leq \left(\frac{2ab}{a+b}\right)^{\vartheta(b)},$$

where the functions are

$$\theta(b) = \frac{\ln b}{2\ln\left(\frac{1+b}{2}\right)} \quad \text{and} \quad \vartheta(b) = \frac{\ln b}{2\ln\left(\frac{2b}{1+b}\right)}.$$

Theorem 2. *For $1 \leq b \leq a$, we have*

$$\left(\frac{a+b}{2}\right)^{\theta(a)} \leq \sqrt{ab} \leq \left(\frac{2ab}{a+b}\right)^{\vartheta(a)},$$

where the functions are

$$\theta(a) = \frac{\ln a}{2\ln\left(\frac{1+a}{2}\right)} \quad \text{and} \quad \vartheta(a) = \frac{\ln a}{2\ln\left(\frac{2a}{1+a}\right)}.$$

In Theorems 1 and 2, we give the double inequality for the case of (I) $0 < b \leq a \leq 1$, (II) $1 \leq b \leq a$. Next, in Theorems 3 and 4, we establish the double inequalities for the case of (I) $1 \leq a < 2$ and $0 < b \leq \frac{1}{2}$, (II) $a \geq 2$ and $0 < b \leq \frac{1}{2}$ which are derived from the left hand side of the inequality (0.1) as follows.

Theorem 3. *If $1 \leq a < 2$ and $0 < b \leq \frac{1}{2}$, then we have*

$$\left(\frac{2ab}{a+b}\right)^{\theta} \leq \sqrt{ab} \leq \left(\frac{2ab}{a+b}\right)^{\vartheta},$$

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where the constants

$$\theta = \frac{\ln 2}{2\ln\left(\frac{3}{2}\right)} \cong 0.854756 \quad \textit{and} \quad \vartheta = 0$$

are the best possible.

Theorem 4. *If $a \geq 2$ and $0 < b \leq \frac{1}{2}$, then we have*

$$\left(\frac{2ab}{a+b}\right)^{\theta} \leq \sqrt{ab} \leq \left(\frac{2ab}{a+b}\right)^{\vartheta(a)},$$

where the constant $\theta = \frac{1}{2}$ is the best possible and the function

$$\vartheta(a) = \frac{\ln\left(\frac{a}{2}\right)}{2\ln\left(\frac{a}{a+\frac{1}{2}}\right)}.$$

From Theorem 1 and 2, we obtain double inequalities as follows.

Corollary 5. *For $a \geq 1$ and $b \geq 1$, we have*

$$\left(\frac{a+b}{2}\right)^{\theta} \leq \sqrt{ab} \leq \left(\frac{a+b}{2}\right)^{\vartheta},$$

where the constants $\theta = \frac{1}{2}$ and $\vartheta = 1$ are the best possible.

Corollary 6. *For $0 < a \leq 1$ and $0 < b \leq 1$, we have*

$$\left(\frac{2ab}{a+b}\right)^{\theta} \leq \sqrt{ab} \leq \left(\frac{2ab}{a+b}\right)^{\vartheta},$$

where the constants $\theta = 1$ and $\vartheta = \frac{1}{2}$ are the best possible.

2. Proof of main results

2.1. Proof of Theorem 1 and Theorem 2

We show four lemmas to prove Theorem 1 and Theorem 2.

Lemma 7. *For $0 < b \leq a \leq 1$, we have*

$$\frac{\ln b}{2\ln\left(\frac{b+1}{2}\right)} \geq \frac{\ln\sqrt{ab}}{\ln\left(\frac{a+b}{2}\right)}.$$

Proof of Lemma 7. For fixed b, let us denote

$$f(a) = \frac{\ln\sqrt{ab}}{\ln\left(\frac{a+b}{2}\right)}.$$

Then we have derivative

$$f'(a) = \frac{a\ln\left(\frac{a+b}{2}\right) + b\ln\left(\frac{a+b}{2}\right) - 2a\ln\left(\sqrt{ab}\right)}{2a(a+b)\left(\ln\left(\frac{a+b}{2}\right)\right)^2}$$

and by the right hand side of the inequality (0.1),

$$\begin{aligned}
&a\ln\left(\frac{a+b}{2}\right) + b\ln\left(\frac{a+b}{2}\right) - 2a\ln\left(\sqrt{ab}\right) \\
&\geq a\ln\left(\sqrt{ab}\right) + b\ln\left(\sqrt{ab}\right) - 2a\ln\left(\sqrt{ab}\right) \\
&= -(a-b)\ln\left(\sqrt{ab}\right) \\
&\geq 0.
\end{aligned}$$

From $f'(a) > 0$ for $b < a < 1$, $f(a)$ is strictly increasing for $b < a < 1$. Hence, we can get

$$f(a) \le f(1) = \frac{\ln b}{2\ln\left(\frac{b+1}{2}\right)}.$$

□

Lemma 8. *For* $1 \le b \le a$, *we have*

$$\frac{\ln\sqrt{ab}}{\ln\left(\frac{a+b}{2}\right)} \ge \frac{\ln a}{2\ln\left(\frac{a+1}{2}\right)}.$$

Proof of Lemma 8. For fixed a, let us denote

$$f(b) = \frac{\ln\sqrt{ab}}{\ln\left(\frac{a+b}{2}\right)}.$$

Then we have derivative

$$f'(b) = \frac{a\ln\left(\frac{a+b}{2}\right) + b\ln\left(\frac{a+b}{2}\right) - 2b\ln\left(\sqrt{ab}\right)}{2b(a+b)\left(\ln\left(\frac{a+b}{2}\right)\right)^2}$$

and by the right hand side of the inequality (0.1),

$$\begin{aligned}
&a\ln\left(\frac{a+b}{2}\right) + b\ln\left(\frac{a+b}{2}\right) - 2b\ln\left(\sqrt{ab}\right) \\
&\ge a\ln\left(\sqrt{ab}\right) + b\ln\left(\sqrt{ab}\right) - 2b\ln\left(\sqrt{ab}\right) \\
&= (a-b)\ln\left(\sqrt{ab}\right) \\
&\ge 0.
\end{aligned}$$

From $f'(b) > 0$ for $1 < b < a$, $f(b)$ is strictly increasing for $1 < b < a$. Hence, we can get

$$f(b) \ge f(1) = \frac{\ln a}{2\ln\left(\frac{a+1}{2}\right)}.$$

□

Lemma 9. *For* $0 < b \le a \le 1$, *we have*

$$\frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)} \ge \frac{\ln b}{2\ln\left(\frac{2b}{b+1}\right)}.$$

Proof of Lemma 9. For fixed b, let us denote

$$f(a) = \frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)}.$$

Then we have derivative

$$f'(a) = \frac{a\ln\left(\frac{2ab}{a+b}\right) + b\ln\left(\frac{2ab}{a+b}\right) - 2b\ln\left(\sqrt{ab}\right)}{2a(a+b)\left(\ln\left(\frac{2ab}{a+b}\right)\right)^2}$$

and by the left hand side of the inequality (0.1),

$$\begin{aligned}
&a\ln\left(\frac{2ab}{a+b}\right)+b\ln\left(\frac{2ab}{a+b}\right)-2b\ln\left(\sqrt{ab}\right)\\
&\le a\ln\left(\sqrt{ab}\right)+b\ln\left(\sqrt{ab}\right)-2b\ln\left(\sqrt{ab}\right)\\
&=(a-b)\ln\left(\sqrt{ab}\right)\\
&\le 0\,.
\end{aligned}$$

From $f'(a)<0$ for $b<a<1$, $f(a)$ is strictly decreasing for $b<a<1$. Hence, we can get

$$f(a)\ge f(1)=\frac{\ln b}{2\ln\left(\frac{2b}{b+1}\right)}\,.$$

□

Lemma 10. *For $1\le b\le a$, we have*

$$\frac{\ln a}{2\ln\left(\frac{2a}{a+1}\right)}\ge\frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)}\,.$$

Proof of Lemma 10. For fixed a, let us denote

$$f(b)=\frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)}\,.$$

Then we have derivative

$$f'(b)=\frac{-2a\ln\left(\sqrt{ab}\right)+a\ln\left(\frac{2ab}{a+b}\right)+b\ln\left(\frac{2ab}{a+b}\right)}{2b(a+b)\left(\ln\left(\frac{2ab}{a+b}\right)\right)^2}$$

and by the left hand side of the inequality (0.1),

$$\begin{aligned}
&a\ln\left(\frac{2ab}{a+b}\right)+b\ln\left(\frac{2ab}{a+b}\right)-2a\ln\left(\sqrt{ab}\right)\\
&\le a\ln\left(\sqrt{ab}\right)+b\ln\left(\sqrt{ab}\right)-2a\ln\left(\sqrt{ab}\right)\\
&=-(a-b)\ln\left(\sqrt{ab}\right)\\
&\le 0\,.
\end{aligned}$$

From $f'(b)<0$ for $1<b<a$, $f(b)$ is strictly decreasing for $1<b<a$. Hence, we can get

$$f(b)\le f(1)=\frac{\ln a}{2\ln\left(\frac{2a}{a+1}\right)}\,.$$

□

Proof of Theorem 1. By lemma 7 and 9 for $0<b\le a\le 1$, we have

$$\left(\frac{\ln b}{2\ln\left(\frac{b+1}{2}\right)}\right)\ln\left(\frac{a+b}{2}\right)\le\ln\sqrt{ab}\le\left(\frac{\ln b}{2\ln\left(\frac{2b}{b+1}\right)}\right)\ln\left(\frac{2ab}{a+b}\right).$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. By lemma 8 and 10 for $1 \leq b \leq a$, we have

$$\left(\frac{\ln a}{2\ln\left(\frac{a+1}{2}\right)}\right)\ln\left(\frac{a+b}{2}\right) \leq \ln\sqrt{ab} \leq \left(\frac{\ln a}{2\ln\left(\frac{2a}{a+1}\right)}\right)\ln\left(\frac{2ab}{a+b}\right).$$

The proof of Theorem 2 is complete. □

2.2 Proof of Theorem 3 and Theorem 4

Proof of Theorem 3. For fixed b, let us denote

$$f(a) = \frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)},$$

then we have derivative

$$f'(a) = \frac{f_1(a)}{2a(a+b)\left(\ln\left(\frac{2ab}{a+b}\right)\right)^2},$$

where $f_1(a) = a\ln\left(\frac{2ab}{a+b}\right) + b\ln\left(\frac{2ab}{a+b}\right) - b\ln(ab)$. By $f_1'(a) = \ln\left(\frac{2ab}{a+b}\right) < 0$, $f_1(a)$ is strictly decreasing for a. From the left hand side of the inequality (0.1), we have

$$\begin{aligned} f_1(a) &< f_1(1) \\ &\leq \frac{1}{2}\ln b + \frac{b}{2}\ln b - b\ln b \\ &= \frac{1-b}{2}\ln b < 0 \end{aligned}$$

for $0 < b \leq \frac{1}{2}$. Thus, $f'(a) < 0$ for a and $f(a)$ is strictly decreasing for a. Hence, we can get

$$f(2) < f(a) \leq f(1)$$

for $1 \leq a < 2$. Here, we set

$$g(b) = f(2) = \frac{\ln(2b)}{2\ln\left(\frac{4b}{b+2}\right)} \quad \textit{and} \quad h(b) = f(1) = \frac{\ln b}{2\ln\left(\frac{2b}{1+b}\right)}.$$

The derivative of $g(b)$ gives

$$g'(b) = \frac{g_1(b)}{2b(b+2)\left(\ln\left(\frac{4b}{b+2}\right)\right)^2},$$

where $g_1(b) = -2\ln(2b) + b\ln\left(\frac{4b}{b+2}\right) + 2\ln\left(\frac{4b}{b+2}\right)$. By $g_1'(b) = \ln\left(\frac{4b}{b+2}\right) < 0$, $g_1(b)$ is strictly decreasing for $0 < b < \frac{1}{2}$. From $g_1(b) < \lim_{b\to 0^+} g_1(b) = 0$ and $g'(b) < 0$ for $0 < b < \frac{1}{2}$, $g(b)$ is strictly decreasing for $0 < b < \frac{1}{2}$. Hence, we can get

$$g(b) \geq g\left(\frac{1}{2}\right) = 0$$

for $0 < b \leq \frac{1}{2}$. The derivative of $h(b)$ gives

$$h'(b) = \frac{h_1(b)}{2b(b+1)\left(\ln\left(\frac{2b}{b+1}\right)\right)^2},$$

where $h_1(b) = -\ln b + b \ln\left(\frac{2b}{b+1}\right) + \ln\left(\frac{2b}{b+1}\right)$. By $h_1'(b) = \ln\left(\frac{2b}{1+b}\right) < 0$, $h_1(b)$ is strictly decreasing for $0 < b < \frac{1}{2}$. From $h_1(b) > h_1(\frac{1}{2}) = -\frac{3}{2}\ln\left(\frac{3}{2}\right) + \ln 2 \cong 0.0849495$ and $h'(b) > 0$ for $0 < b < \frac{1}{2}$, $h(b)$ is strictly increasing for $0 < b < \frac{1}{2}$. Hence, we can get

$$h(b) \leq h\left(\frac{1}{2}\right) = \frac{\ln 2}{2\ln\left(\frac{3}{2}\right)}$$

for $0 < b \leq \frac{1}{2}$. Therefore, we obtain

$$0 < \frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)} \leq \frac{\ln 2}{2\ln\left(\frac{3}{2}\right)}$$

for $1 \leq a < 2$ and $0 < b \leq \frac{1}{2}$. The proof of Theorem 3 is complete. □

Proof of Theorem 4. For fixed b, let us denote

$$f(a) = \frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)}.$$

Since $f(a)$ is strictly decreasing for a, we have $f(a) < f(2)$. Here, we set

$$g(b) = f(2) = \frac{\ln(2b)}{2\ln\left(\frac{4b}{b+2}\right)}.$$

Since $g(b)$ is strictly decreasing for $0 < b < \frac{1}{2}$, we can get

$$g(b) < \lim_{b \to 0^+} g(b) = \frac{1}{2}$$

for $0 < b \leq \frac{1}{2}$. We set

$$h(b) = \frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)},$$

then the derivative of $h(b)$ is

$$h'(b) = \frac{f_1(b)}{2b(a+b)\left(\ln\left(\frac{2ab}{a+b}\right)\right)^2},$$

where $f_1(b) = a\ln\left(\frac{2ab}{a+b}\right) + b\ln\left(\frac{2ab}{a+b}\right) - a\ln(ab)$. By $f_1'(b) = \ln\left(\frac{2ab}{a+b}\right) < 0$, $f_1(b)$ is strictly decreasing for $0 < b < \frac{1}{2}$. From $f_1(b) < \lim_{b\to 0^+} f_1(b) = -a\ln\left(\frac{a}{2}\right) < 0$ for $a > 2$, $h'(b) < 0$ and $h(b)$ is strictly decreasing for $0 < b < \frac{1}{2}$. Hence, we can get

$$h(b) \geq h\left(\frac{1}{2}\right) = \frac{\ln\left(\frac{a}{2}\right)}{2\ln\left(\frac{a}{a+\frac{1}{2}}\right)}$$

for $0 < b \leq \frac{1}{2}$. Therefore, we obtain

$$\frac{\ln\left(\frac{a}{2}\right)}{2\ln\left(\frac{a}{a+\frac{1}{2}}\right)} \leq \frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)} < \frac{1}{2}.$$

for $a \geq 2$ and $0 < b \leq \frac{1}{2}$. The proof of Theorem 4 is complete. □

2.3 Proof of Corollary 5 and Corollary 6

Proof of Corollary 5. We assume $1 \leq b \leq a$ then, let us denote

$$f(b) = \frac{\ln \sqrt{ab}}{\ln \left(\frac{a+b}{2}\right)}$$

for fixed a. By Lemma 8, we have

$$f(b) = \frac{\ln \sqrt{ab}}{\ln \left(\frac{a+b}{2}\right)} \geq \frac{\ln a}{2\ln \left(\frac{a+1}{2}\right)} = g(a)\,.$$

Here, the derivative of $g(a)$ gives

$$g'(a) = \frac{h(a)}{2a(a+1)\left(\ln \left(\frac{a+1}{2}\right)\right)^2}\,,$$

where

$$h(a) = a\ln \left(\frac{a+1}{2}\right) + \ln \left(\frac{a+1}{2}\right) - a\ln a$$

and we have

$$h'(a) = \ln \left(\frac{a+1}{2}\right) - \ln a$$

From $0 < \frac{a+1}{2a} < 1$, $h'(a) < 0$ and $h(a)$ is strictly decreasing for $a > 1$. By $h(1) = 0$, we have $h(a) < 0$ and $g'(a) < 0$ for $a > 1$. Therefore, $g(a)$ is strictly decreasing for $a > 1$ and $g(a) > \lim_{a\to\infty} g(a) = \frac{1}{2}$. By Lemma 8, $f(b)$ is strictly increasing for $1 < b < a$ and we obtain $f(b) \leq f(a) = 1$. Thus, we obtain

$$\frac{1}{2} < \frac{\ln \sqrt{ab}}{\ln \left(\frac{a+b}{2}\right)} \leq 1\,,$$

where the constants $\frac{1}{2}$ and 1 are the best possible. The proof of Corollary 5 is complete. □

Proof of Corollary 6. We assume $0 < b \leq a \leq 1$ then, let us denote

$$f(a) = \frac{\ln \sqrt{ab}}{\ln \left(\frac{2ab}{a+b}\right)}$$

for fixed b. By Lemma 9, we have

$$f(a) = \frac{\ln \sqrt{ab}}{\ln \left(\frac{2ab}{a+b}\right)} \geq \frac{\ln b}{2\ln \left(\frac{2b}{b+1}\right)} = g(b)\,.$$

Here, the derivative of $g(b)$ gives

$$g'(b) = \frac{h(b)}{2b(b+1)\left(\ln \left(\frac{2b}{b+1}\right)\right)^2}\,,$$

where

$$h(b) = b\ln \left(\frac{2b}{b+1}\right) + \ln \left(\frac{2b}{b+1}\right) - \ln b$$

and we have

$$h'(b) = \ln \left(\frac{2b}{b+1}\right)$$

From $0 < \frac{2b}{b+1} < 1$, $h'(b) < 0$ and $h(b)$ is strictly decreasing for $0 < b < 1$. By $h(1) = 0$, we have $h(b) > 0$ and $g'(b) > 0$ for $0 < b < 1$. Therefore, $g(b)$ is strictly increasing for $0 < b < 1$ and

$g(b) > \lim_{b\to 0} g(b) = \frac{1}{2}$. By Lemma 9, $f(a)$ is strictly decreasing for $b < a < 1$ and we obtain $f(a) \leq f(b) = 1$. Thus, we obtain

$$\frac{1}{2} < \frac{\ln\sqrt{ab}}{\ln\left(\frac{2ab}{a+b}\right)} \leq 1\,,$$

where the constants $\frac{1}{2}$ and 1 are the best possible. The proof of Corollary 6 is complete. □

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