An elementary proof of some inequality derived from the function $(b^x - a^x)/x$

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Abstract: In this article, we prove that the inequality $(x+\gamma)\,(a+b/2)^{\gamma}/x > (b^{x+\gamma} - a^{x+\gamma})/(b^x - a^x)$ holds for $0 < \gamma < 1$, $0 < x < 1$ and $0 < a < b$ using by elementary method.

Keywords: inequality, monotonically decreasing function, monotonically increasing function.

1. Introduction

Qi Feng et al.[1,2,3,4,5,6] studied some properties and inequalities derived from the function $(b^x - a^x)/x$. Qi Feng et al.[3] and Z. Liu [5] proved that the inequality

$$\frac{b^{x+\gamma} - a^{x+\gamma}}{b^x - a^x} \geq \frac{x+\gamma}{x}\left(\frac{a+b}{2}\right)^{\gamma} \tag{0.1}$$

holds for $\gamma \geq 1$, $x \geq 1$ and $0 < a < b$. They [3] showed the inequality

$$\frac{a+b}{2} \leq \frac{1}{b-a}\int_a^b t^{\gamma}dt = \frac{b^{1+\gamma} - a^{1+\gamma}}{(b-a)(1+\gamma)} \tag{0.2}$$

holds for $\gamma \geq 1$ and $0 < a < b$ and the inequality

$$\frac{x(b^{x+\gamma} - a^{x+\gamma})}{(x+\gamma)(b^x - a^x)} \geq \frac{b^{1+\gamma} - a^{1+\gamma}}{(1+\gamma)(b-a)} \tag{0.3}$$

holds for $x \geq 1$, $\gamma \geq 0$ and $0 < a < b$. The above inequalities (0.2) and (0.3) are important roles to prove the inequality (0.1) in [3]. Since the inequality (0.2) does not hold for $0 < \gamma < 1$, they [3] can not prove that the inequality (0.1) holds for $0 < \gamma < 1$ or $0 < x < 1$.

In this shote note, we shall give a directly proof of the reversed inequality (0.1) for $0 < \gamma < 1$ and $0 < x < 1$ in an elementary method.

Main Theorem. The inequality

$$\frac{x+\gamma}{x}\left(\frac{a+b}{2}\right)^{\gamma} > \frac{b^{x+\gamma} - a^{x+\gamma}}{b^x - a^x} \tag{0.4}$$

holds for all $0 < \gamma < 1$, $0 < x < 1$ and $0 < a < b$.

2. Preliminaries and proof of Main Theorem

We need the following lemmas to prove Main Theorem.

Lemma 2.1. If $\gamma > 0$, $x \geq 0$ and $0 < t < 1$, then

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$$1 - t^{\gamma} + \gamma t^{x+\gamma} \ln t > 0\,.$$

Proof. We set

$$f(\gamma, t, x) = 1 - t^{\gamma} + \gamma t^{x+\gamma} \ln t\,.$$

Since the partial derivative

$$\frac{\partial}{\partial x} f(\gamma, t, x) = \gamma t^{x+\gamma} (\ln t)^2 > 0\,,$$

$f(\gamma, t, x)$ is strictly increasing for $x > 0$. Thus, $f(\gamma, t, x) > f(\gamma, t, 0)$. Since

$$f(\gamma, t, 0) = 1 - t^{\gamma} + \gamma t^{\gamma} \ln t$$

and the partial derivative

$$\frac{\partial}{\partial t} f(\gamma, t, 0) = \gamma^2 t^{-1+\gamma} \ln t < 0\,,$$

$f(\gamma, t, 0)$ is strictly decreasing for $0 < t < 1$. Since $f(\gamma, t, 0) > f(\gamma, 1, 0) = 0$, we can get $f(\gamma, t, x) > 0$ for all $\gamma > 0$, $x \geq 0$ and $0 < t < 1$. □

Lemma 2.2. If $\gamma > 0$, $0 < x < 1$ and $0 < t < 1$, then

$$\frac{(x + \gamma)(1 - t^{x})}{x(1 - t^{x+\gamma})} > \frac{(1 + \gamma)(1 - t)}{1 - t^{1+\gamma}}\,.$$

Proof. We set

$$f(\gamma, t, x) = \frac{(x + \gamma)(1 - t^{x})}{x(1 - t^{x+\gamma})}\,.$$

Then the partial derivative

$$\frac{\partial}{\partial x} f(\gamma, t, x) = -\frac{g(\gamma, t, x)}{(1 - t^{x+\gamma})^2 x^2},$$

where

$$g(\gamma, t, x) = \gamma - \gamma t^{x} - \gamma t^{x+\gamma} + \gamma t^{2x+\gamma} + \gamma t^{x} x \ln t - \gamma t^{x+\gamma} x \ln t + t^{x} x^2 \ln t - t^{x+\gamma} x^2 \ln t\,.$$

Then the partial derivative

$$\frac{\partial}{\partial x} g(\gamma, t, x) = h(\gamma, t, x) t^{x} \ln t,$$

where

$$h(\gamma, t, x) = -2\gamma t^{\gamma} + 2\gamma t^{x+\gamma} + 2x - 2t^{\gamma} x + \gamma x \ln t - \gamma t^{\gamma} x \ln t + x^2 \ln t - t^{\gamma} x^2 \ln t\,.$$

Then the partial derivatives

$$\frac{\partial}{\partial x} h(\gamma, t, x) = 2 - 2t^{\gamma} + \gamma \ln t - \gamma t^{\gamma} \ln t + 2\gamma t^{x+\gamma} \ln t + 2x \ln t - 2t^{\gamma} x \ln t$$

and

$$\frac{\partial^2}{\partial x^2}h(\gamma,t,x) = 2(1 - t^\gamma + \gamma t^{x+\gamma}\ln t)\ln t\,.$$

Therefore, by Lemma 2.1, we have

$$\frac{\partial^2}{\partial x^2}h(\gamma,t,x) < 0$$

for all $\gamma > 0$, $x > 0$ and $0 < t < 1$. Thus, $\partial h(\gamma,t,x)/\partial x$ is strictly decreasing for $x > 0$. Then we have

$$\frac{\partial}{\partial x}h(\gamma,t,0) > \frac{\partial}{\partial x}h(\gamma,t,x)\,,$$

$$\frac{\partial}{\partial x}h(\gamma,t,0) = 2 - 2t^\gamma + \gamma\ln t + \gamma t^\gamma \ln t$$

and

$$\frac{\partial^2}{\partial x\partial\gamma}h(\gamma,t,0) = (1 - t^\gamma + \gamma t^\gamma\ln t)\ln t\,.$$

By Lemma 2.1, we have

$$\frac{\partial^2}{\partial x\partial\gamma}h(\gamma,t,0) < 0.$$

Therefore, $\partial h(\gamma,t,0)/\partial x$ is strictly decreasing for $\gamma > 0$. Since $\partial h(0,t,0)/\partial x = 0$, $\partial h(\gamma,t,0)/\partial x < 0$ for all $\gamma > 0$ and $0 < t < 1$. Thus, $\partial h(\gamma,t,x)/\partial x < 0$ and $h(\gamma,t,x)$ is strictly decreasing for $x > 0$. Since $h(\gamma,t,0) = 0$, we have $h(\gamma,t,x) < 0$ for all $\gamma > 0$, $x > 0$ and $0 < t < 1$. Therefore, $\partial g(\gamma,t,x)/\partial x > 0$ and $g(\gamma,t,x)$ is strictly increasing for $x > 0$. Since $g(\gamma,t,0) = 0$, we get $g(\gamma,t,x) > 0$ and $\partial f(\gamma,t,x)/\partial x < 0$. Thus, $f(\gamma,t,x)$ is strictly decreasing for $x > 0$. Since $f(\gamma,t,x) > f(\gamma,t,1)$, we completed the proof of Lemma 2.2. □

Lemma2.3. If $0 < \gamma < 1$ and $0 < t < 1$, then

$$\frac{(1+\gamma)(1-t)}{1-t^{1+\gamma}}\left(\frac{1+t}{2}\right)^\gamma > 1\,.$$

Proof. We set

$$f(\gamma,t) = \frac{(1+\gamma)(1-t)}{1-t^{1+\gamma}}\left(\frac{1+t}{2}\right)^\gamma\,.$$

Then the partial derivative

$$\frac{\partial}{\partial t}f(\gamma,t) = \frac{-g(\gamma,t)}{2^\gamma(1+t)^{1-\gamma}(1-t^{1+\gamma})^2},$$

where $g(\gamma,t) = (1+\gamma)(1-\gamma+t+\gamma t - t^\gamma - \gamma t^\gamma - t^{1+\gamma} + \gamma t^{1+\gamma})$. Then we have the partial derivative

$$\frac{\partial}{\partial t}g(\gamma,t) = \frac{(1+\gamma)\,h(\gamma,t)}{t},$$

where $h(\gamma,t) = t - \gamma t^\gamma - t^{1+\gamma} + \gamma t^{1+\gamma}$. Then we have the partial derivative

$$\frac{\partial}{\partial \gamma} h(\gamma, t) = t^{\gamma} k(\gamma, t),$$

where $k(\gamma, t) = -1 + t - \gamma \ln t - t \ln t + \gamma t \ln t$. Since the partial derivative

$$\frac{\partial}{\partial \gamma} k(\gamma, t) = (-1 + t) \ln t > 0\,,$$

$k(\gamma, t)$ is strictly increasing for $0 < \gamma < 1$. Since $k(0, t) = -1 + t - t \ln t < 0$ and $k(1, t) = -1 + t - \ln t > 0$, there exists $\gamma(t)$ such that $0 < \gamma(t) < 1$ and $k(\gamma(t), t) = 0$. Since $k(\gamma, t) < 0$ for all $0 < \gamma < \gamma(t)$ and $k(\gamma, t) > 0$ for all $\gamma(t) < \gamma < 1$, $h(\gamma, t)$ is strictly decreasing for $0 < \gamma < \gamma(t)$ and $h(\gamma, t)$ is strictly increasing for $\gamma(t) < \gamma < 1$. Since $h(0, t) = 0$ and $h(1, t) = 0$, we have $h(\gamma, t) < 0$ for all $0 < \gamma < 1$ and $0 < t < 1$. Therefore, $\partial g(\gamma, t)/\partial t < 0$. Thus, $g(\gamma, t)$ is strictly decreasing for $0 < t < 1$. Since $g(\gamma, 1) = 0$, we get $g(\gamma, t) > 0$. Hence $\partial f(\gamma, t)/\partial t < 0$ and $f(\gamma, t)$ is strictly decreasing for $0 < t < 1$. Therefore, we have $f(\gamma, t) > \lim_{t \to 1} f(\gamma, t) = 1$. □

Proof of Main Theorem. We assume that $t = a/b$. Then the inequality (0.4) is equivalent to

$$\frac{(x + \gamma)(1 - t^x)}{x(1 - t^{x+\gamma})} \left(\frac{1 + t}{2} \right)^{\gamma} > 1,$$

where $0 < \gamma < 1$, $0 < x < 1$ and $0 < t < 1$. By Lemmas 2.2 and 2.3, the proof of Main Theorem is completed. □

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