Global Well-Posedness and Exponential Attractor for the Oregonator System with Global Feedback

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Abstract. The Belousov-Zhabotinskii (BZ) reaction is a phenomenon of a nonlinear chemical oscillator. The Oregonator model system with photochemical pathway is a mathematical model of the photosensitive BZ reaction. In the paper, the global well-posedness of this system is shown and to investigate the large time behavior of the solutions the exponential attractor is constructed.

Key words: Reaction-Diffusion System, Infinite-Dimensional Dynamical System, Exponential Attractor.

1. Introduction.

In this paper we consider the following Oregonator model system of equations with global feedback:

$$
\text{(OR)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = D_u \Delta u + \dfrac{1}{\varepsilon}\left\{u(1-u) - (fv+\phi_u)\dfrac{u-q}{u+q}\right\} & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = D_v \Delta v + u - v \quad \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x) \quad \text{in} \quad \Omega, \end{cases}
$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$. The unknown functions u and v describe $HBrO_2$ and the Ru(III) catalyst concentrations, respectively. The coefficients D_u, D_v, ε, f and q are positive constants. We denote the feedback term by ϕ_u. In this paper we define ϕ_u as

$$
\phi_u(t) = a \int_\Omega u(x,t)^p \, dx + b,
$$

where $a \geq 0$ is a gain constant, $b \in \mathbb{R}$ is an offset constant and $p \geq 1$ is an arbitrary number. This feedback function is introduced in [4] (cf. [5]). If $a = b = 0$, then (OR) is a two-variable Oregonator system (see [2, 8]). The system (OR) is a modified model to include the photosensitivity of the BZ reaction [3]. The system has presented many physical and mathematical phenomena, which have been studied in both sides. Indeed, a stabilized chemical packet can be seen by numerical computation in [4]. However, from the mathematical point of view the well-posedness is not treated as far as the authors know even if the feedback term is absent ($a = b = 0$). The aim of this paper is to show the global well-posedness of (OR) and to investigate the behavior of solutions. Here the well-posedness includes the time global existence, uniqueness and the continuous dependence of the solution upon the data.

The following theorem is concerned with the global well-posedness of (OR).

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Theorem 1.1 (Global Well-Posedness). *Let* $(u_0, v_0) \in H^2_N(\Omega) \times H^2_N(\Omega)$ *with* $u_0 \in [q, 1]$ *and* $v_0 \in [0, 1]$. *Then there exists a unique solution* (u, v) *for* (OR) *satisfying*

$$u,\ v \in \mathcal{C}([0, \infty);\ H^2_N(\Omega)) \cap \mathcal{C}^1((0, \infty);\ H^1(\Omega)) \cap \mathcal{C}((0, \infty);\ H^3_N(\Omega)).$$

In addition, u *and* v *satisfy* $u \in [q, 1]$ *and* $v \in [0, 1]$.

Hence, the solution map $S(t) : (u_0, v_0) \mapsto (u(t), v(t))$ *generates a continuous dynamical system in*

$$K = \{(u_0, v_0) \in H^2_N \times H^2_N \mid u_0 \in [q, 1],\ v_0 \in [0, 1]\}.$$

Here, we define

$$H^m_N(\Omega) = \left\{ u \in H^m(\Omega) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\} \quad \text{for} \quad m = 2, 3.$$

A priori uniform estimates obtained in above admit also existence of absorbing set $\mathcal{B}$. Here, $\mathcal{B}$ is an *absorbing set* if $\mathcal{B}$ is a compact subset of the phase space and, for every bounded subset $B \subset K$, there is a time t_B which may depend on B such that $\bigcup_{t \geq t_B} S(t)B \subset \mathcal{B}$. By the existence of absorbing set we can construct a global attractor. In fact, according to Temam [7, Theorem 1.1], the ω-limit set $\mathcal{A}$ of $\mathcal{B}$ defined by

$$\mathcal{A} = \omega(\mathcal{B}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\mathcal{B}}$$

is the global attractor for $(\{S(t)\}_{t \geq 0}, K)$. Here, $(\{S(t)\}_{t \geq 0}, K)$ means the dynamical system generated by $S(t)$ in K.

By constructing an exponential attractor, we can obtain more precise informations for the behavior of solutions. The concept of an exponential attractor is established by Eden, Foias, Nicolaenko and Temam [1]. Let us assume that H is a separable Hilbert space, A is a positive definite self-adjoint linear operator in H, the inverse of which is a compact operator on H. We define the set $\mathcal{X} = \overline{\bigcup_{t \geq t_{\mathcal{B}}} S(t)\mathcal{B}}$ with fixed $t_{\mathcal{B}}$ such that $\bigcup_{t \geq t_{\mathcal{B}}} S(t)\mathcal{B} \subset \mathcal{B}$. It is easily observed that $\mathcal{X}$ is a compact subset of H such that $\mathcal{A} \subset \mathcal{X} \subset \mathcal{B}$ and $\mathcal{X}$ is absorbing and invariant for $(\{S(t)\}_{t \geq 0}, K)$. Therefore, to know the large time behavior of solution it suffices to consider $(\{S(t)\}_{t \geq 0}, \mathcal{X})$.

The exponential attractor is defined as follows, see Eden et al. [1].

Definition. A subset $\mathcal{M} \subset \mathcal{X}$ is called the *exponential attractor* for $(\{S(t)\}_{t \geq 0}, \mathcal{X})$ if the following conditions are satisfied,

i) $\mathcal{A} \subset \mathcal{M} \subset \mathcal{X}$;

ii) $\mathcal{M}$ is a compact subset of H and is an invariant set for $S(t)$;

iii) $\mathcal{M}$ has finite fractal dimension $d_F(\mathcal{M})$;

iv) $h(S(t)\mathcal{X}, \mathcal{M}) \leq c_0 \exp(-c_1 t)$ for $t \geq 0$ with some constants $c_0,\ c_1 > 0$, where

$$h(B_0, B_1) = \sup_{U \in B_0} \inf_{V \in B_1} \|U - V\|_H$$

denotes the Hausdorff pseudodistance of two sets B_0 and B_1.

In the following theorem, we construct the exponential attractor for the dynamical system of (OR).

Theorem 1.2 (Exponential Attractor). *Let* B *be an arbitrary bounded set in* K. *Then, there exist a time* t_B *and a universal constant* R *for* B *such that*

$$\sup_{(u_0, v_0) \in B} \|S(t)(u_0, v_0)\|_{H^2 \times H^2} \leq R \quad \text{for all} \quad t \geq t_B.$$

Therefore, a compact set $\mathcal{B} = \{(u, v) \in K;\ \|(u, v)\|_{H^2 \times H^2} \leq R\}$ *in* $H^1(\Omega) \times H^1(\Omega)$ *is an absorbing set for the dynamical system* $(\{S(t)\}_{t \geq 0}, K)$.

We denote by $t_{\mathcal{B}}$ a number satisfying $S(t)\mathcal{B} \subset \mathcal{B}$ for $t \geq t_{\mathcal{B}}$ and we set

$$\mathcal{X} = \overline{\bigcup_{t \geq t_{\mathcal{B}}} S(t)\mathcal{B}}^{H^1 \times H^1}.$$

Then the dynamical system $(\{S(t)\}_{t\geq 0}, \mathcal{X})$ admits an exponential attractor $\mathcal{M}$.

We devote Section 2 to list the preliminary results. In Section 3 we construct the time local solution by using semigroup method for the abstract semilinear parabolic equation. Section 4 is devoted to give a priori estimates which also admits the existence of absorbing set. We construct the exponential attractor in Section 5.

Remark. In the case of the periodic boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_u \Delta u + \dfrac{1}{\varepsilon}\left\{u(1-u) - (fv + \phi_u)\dfrac{u-q}{u+q}\right\} \quad \text{in} \quad (\mathbb{T}_{l_1} \times \mathbb{T}_{l_2}) \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = D_v \Delta v + u - v \quad \text{in} \quad (\mathbb{T}_{l_1} \times \mathbb{T}_{l_2}) \times (0, \infty), \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x) \quad \text{in} \quad \mathbb{T}_{l_1} \times \mathbb{T}_{l_2}, \end{cases}$$

the same results as the main theorems can be also proved. Here $\mathbb{T}_l$ means the one-dimensional torus of period l, i.e., $\mathbb{T}_l = l\mathbb{T} = l\mathbb{R}/\mathbb{Z}$. In this case, we can characterize the fractional power of $(-\Delta + 1)$ easily by using the Fourier expansion.

2. Preliminaries.

We denote by $L^p(\Omega)$, $1 \leq p < \infty$, the complex-valued L^p space with the norm $\|\cdot\|_{L^p}$. Denote also by $H^m(\Omega)$, $m = 0, 1, 2, \cdots$, the complex-valued Sobolev spaces with the norm $\|\cdot\|_{H^m}$. When $m = 0$, $H^0(\Omega) = L^2(\Omega)$. We denote by $\mathcal{B}(\Omega)$ and $\mathcal{C}(\Omega)$ the spaces of complex-valued bounded functions and continuous functions on Ω with norm $\|\cdot\|_{\mathcal{B}}$ and $\|\cdot\|_{\mathcal{C}}$, respectively. Let I be an interval in $\mathbb{R}$, and H be a Banach space. $\mathcal{C}(I; H)$ and $\mathcal{C}^1(I; H)$ are the space of H-valued continuous functions and continuously differentiable functions on I, respectively. $\mathcal{B}(I; H)$ is the space of H-valued bounded functions on I. Let H_1 and H_2 be two Banach spaces. Then, the product space $H = H_1 \times H_2$ can be considered, and the norm is given by $\|\cdot\|_H = \|\cdot\|_{H_1} + \|\cdot\|_{H_2}$. Especially, we use the notations $\mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$, $\mathbb{H}^1(\Omega) = H^1(\Omega) \times H^1(\Omega)$ and $\mathbb{H}^m_N(\Omega) = H^m_N(\Omega) \times H^m_N(\Omega)$, $m = 2, 3$. Here the function space $H^m_N(\Omega)$ is defined as

$$H^m_N(\Omega) = \left\{u \in H^m(\Omega) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}, \quad m = 2, 3.$$

For $0 \leq s_0 \leq s \leq s_1 < \infty$,

$$H^s(\Omega) = [H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta, \quad s = (1-\theta)s_0 + \theta s_1,$$

which is the complex interpolation space between $H^{s_0}(\Omega)$ and $H^{s_1}(\Omega)$ with the norm $\|\cdot\|_{H^s}$. The following inequality is true:

$$\|\cdot\|_{H^s} \leq C_\theta \|\cdot\|^{1-\theta}_{H^{s_0}} \|\cdot\|^{\theta}_{H^{s_1}}. \tag{2.1}$$

When $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with

$$\|\cdot\|_{\mathcal{B}} \leq C_s \|\cdot\|_{H^s}. \tag{2.2}$$

When $s=1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $q \in [1,\infty)$ with

$$\|\cdot\|_{L^q} \le C_{p,q} \|\cdot\|_{H^1}^{1-\frac{p}{q}} \|\cdot\|_{L^p}^{\frac{p}{q}}, \tag{2.3}$$

where $1 \le p \le q < \infty$. When $0 \le s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, $\frac{1}{p} = \frac{1-s}{2}$, with

$$\|\cdot\|_{L^p} \le C_s \|\cdot\|_{H^s}. \tag{2.4}$$

From (2.2), (2.3) and (2.4), the inequalities on multiplication of two functions are derived. Let ε be an arbitrary positive number. Then,

$$\|uv\|_{H^m} \le C_{m,\varepsilon} \|u\|_{H^m} \|v\|_{H^{1+\varepsilon}}, \quad u \in H^m(\Omega),\ v \in H^{1+\varepsilon}(\Omega),\ m = 0, 1. \tag{2.5}$$

Consider an operator $A = -D\Delta + 1$, D is positive constant. Then, A is a positive and self-adjoint operator of $L^2(\Omega)$ with the domain $\mathcal{D}(A) = H^2_N(\Omega)$. For $\alpha \ge 0$, the fractional power A^α is defined. The operator A^α is also a positive and self-adjoint operator of $L^2(\Omega)$. It is well-known that

$$\mathcal{D}(A^\alpha) = \begin{cases} H^{2\alpha}(\Omega), & 0 \le \alpha < \frac{3}{4}, \\ H^{2\alpha}_N(\Omega), & \frac{3}{4} < \alpha \le 1 \end{cases}$$

with the norm equivalence. But, we verify that

$$\mathcal{D}(A^\alpha) = H^{2\alpha}_N(\Omega), \quad 1 \le \alpha \le \frac{3}{2}.$$

Indeed, $Au \in \mathcal{D}(A^\alpha)$ shows that $\Delta u \in H^1(\Omega)$ with $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. While $\partial\Omega$ is of class C^3, these then imply that $u \in H^3(\Omega)$.

Next, we give the time local existence theorem for the semilinear abstract parabolic problem:

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & 0 < t \le T, \\ U(0) = U_0 \end{cases} \tag{2.6}$$

in some function space H. It is well-known results that if the nonlinear operator F satisfies appropriate Lipschitz continuity, a local solution can be constructed (e.g. [9]). In [6] the first author and Yagi introduced the modified Lipschitz condition given by (2.9) below.

Proposition 2.1. **([6, Theorem 3.1 and Corollary 3.2])** *For the initial value problem of a semilinear abstract evolution equation* (2.6) *in a Banach space H, assume that A is a closed linear operator of H satisfying that*

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda| + 1}, \quad \lambda \notin \Sigma, \tag{2.7}$$

with $\Sigma = \{\lambda \in \mathbb{C};\, |\arg \lambda| \le \psi\}$, $0 \le \psi < \frac{\pi}{2}$, and $M > 0$ is a constant, and the initial value satisfies an estimate

$$\|A^\alpha U_0\| \le r, \tag{2.8}$$

where $\alpha \in [0,1)$ is some exponent and $r > 0$ is a constant. Assume also that $F(\cdot)$ satisfies a Lipschitz condition

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_H \le\ & P(\|A^\alpha U\|_H + \|A^\alpha \tilde{U}\|_H) \\ & \times \left\{ \|A^\eta (U - \tilde{U})\|_H + (\|A^\eta U\|_H + \|A^\eta \tilde{U}\|_H + 1) \|A^\alpha (U - \tilde{U})\|_H \right\}, \quad U, \tilde{U} \in \mathcal{D}(A^\eta), \end{aligned} \tag{2.9}$$

with an exponent $\eta \in [\alpha, 1)$ and an increasing continuous function $P(\cdot)$. Then, there exists a unique local solution to (2.6) *such that*

$$U \in \mathcal{C}([0,T_r];\, \mathcal{D}(A^\alpha)) \cap \mathcal{C}^1((0,T_r];\, H) \cap \mathcal{C}((0,T_r];\, \mathcal{D}(A)), \quad t^{1-\alpha}U \in \mathcal{B}((0,T_r];\, \mathcal{D}(A)).$$

Moreover, the follwing estimates hold:

$$t^{1-\alpha}\|AU(t)\|_H + \|A^\alpha U(t)\|_H \le C_r, \quad 0 < t \le T_r,$$
$$\sup_{0<t\le T_r} t^{\eta-\alpha}\|A^\eta\{U(t)-V(t)\}\|_H + \max_{0\le t\le T_r}\|A^\alpha\{U(t)-\tilde{U}(t)\}\|_H \le C_r\|A^\alpha\{U_0-\tilde{U}_0\}\|_H,$$

where $\tilde{U}(t)$ is the solution of initial value $\tilde{U}_0 \in \mathcal{D}(A^\alpha)$, and $\tilde{U}_0$ satisfies (2.8).

To construct an exponential attractor let us apply the following proposition:

Proposition 2.2. *Assume that $F(U)$ satisfy the Lipschitz condition*

$$\text{(F)} \qquad \|F(U)-F(\tilde{U})\|_H \le C\|A^{\frac{1}{2}}(U-\tilde{U})\|_H, \quad U, \tilde{U} \in \mathcal{X}.$$

Let $S(t)$ be the solution map of (2.6) *and $\rho \in (0,1]$. Suppose that the mapping $G(t,U_0) = S(t)U_0$ from $[0,T] \times \mathcal{X}$ into $\mathcal{X}$ satisfies the Lipschitz condition*

$$\text{(G)} \qquad \|G(t,U_0)-G(s,\tilde{U}_0)\|_H \le C_T\{\,|t-s|^\rho + \|U_0-\tilde{U}_0\|_H\}, \quad t,s \in [0,T],\ U_0, \tilde{U}_0 \in \mathcal{X},$$

for each $T > 0$. Then, there exists an exponential attractor $\mathcal{M}$ for $(\{S(t)\}_{t\ge0}, \mathcal{X})$.

This proposition follows from a modification similar to [1, Theorem 3.1] which corresponds to the above proposition in the case of $\rho = 1$. The proof is reduced to constructing a similar exponential attractor for a discrete dynamical system $(\{S_*^n\}_{n\ge0}, \mathcal{X})$, where $S_* = S(t_*)$ with a suitable time $t_* > 0$. For the discrete dynamical system, the condition on S_* called the squeezing property plays an important role: for some $\delta \in (0, \frac{1}{4})$, there exists an orthogonal projection P of finite rank N such that for each pair U, $\tilde{U} \in \mathcal{X}$ either $\|S_*U - S_*\tilde{U}\|_H \le \delta\|U-\tilde{U}\|_H$ or $\|(I-P)(S_*U - S_*\tilde{U})\|_H \le \|P(S_*U - S_*\tilde{U})\|_H$. In the case when the dynamical system is determined by a semilinear evolution equation such as (2.6), this property can be verified from the Lipschitz condition (F), see [1, Proposition 3.1]. As a result, the existence of an exponential attractor $\mathcal{M}_*$ for $(\{S_*^n\}_{n\ge0}, \mathcal{X})$ can be obtained, as well the dimension is estimated by $d_F(\mathcal{M}_*) \le N\max\{1, \log(\frac{2L}{\delta}+1)/\log(\frac{1}{4\delta})\}$, where L is a Lipschitz constant of the mapping S_* from $\mathcal{X}$ into itself. Let $\mathcal{M} := G([0,t_*] \times \mathcal{M}_*) = \bigcup_{0\le t\le t_*} S(t)\mathcal{M}_*$. We shall prove that $\mathcal{M}$ is an exponential attractor for $(\{S(t)\}_{t\ge0}, \mathcal{X})$. The conditions i), ii) and iv) in the definition of the exponential attractor can be verified in the same fashion as [1, Theorem 3.1]. Then it suffices to show the finiteness of the fractal dimension $d_F(\mathcal{M})$. Here, we give the claim.

Claim. *Let $\rho \in (0,1]$. Assume that for the normed space $\mathcal{Y}_1$ and $\mathcal{Y}_2$ the map $\psi : \mathcal{Y}_1 \to \mathcal{Y}_2$ is the Hölder continuous, namely,*

$$\|\psi(y)-\psi(\tilde{y})\|_{\mathcal{Y}_2} \le C\|y-\tilde{y}\|_{\mathcal{Y}_1}^\rho, \qquad y, \tilde{y} \in \mathcal{Y}_1.$$

Then for the set $\mathcal{U} \subset \mathcal{Y}_1$ it holds that

$$d_F(\psi(\mathcal{U})) \le \frac{1}{\rho}d_F(\mathcal{U}).$$

If we admit the claim, then the fractal dimension of $\mathcal{M}$ is estimated by $d_F(\mathcal{M}) \le (d_F(\mathcal{M}_*)+1)/\rho < \infty$. Indeed, since

$$\|G(t,U_0)-G(s,\tilde{U}_0)\|_H \le C(|t-s|^\rho + \|U_0-\tilde{U}_0\|_H) \le C(|t-s| + \|U_0-\tilde{U}_0\|_H)^\rho,$$

we have

$$\begin{aligned} d_F(\mathcal{M}) &= d_F(G([0,t_*]\times\mathcal{M}_*)) \\ &\le \frac{1}{\rho} d_F([0,t_*]\times\mathcal{M}_*) \\ &\le \frac{1}{\rho}(d_F(\mathcal{M}_*)+1). \end{aligned}$$

Then the condition iii) is satisfied.

Lastly, we give the proof of the claim. We take the finite ball covering $\{B_\epsilon(x_i)\}_{i=1}^{N_\epsilon(\mathcal{U})}$ of $\mathcal{U}$ such that

$$\mathcal{U} \subset \bigcup_{j=1}^{N_\epsilon(\mathcal{U})} B_\epsilon(x_i), \tag{2.10}$$

where $B_\epsilon(x_i) \subset \mathcal{Y}_1$ is the ball centered at $x_i \in \mathcal{Y}_1$ with the diameter ϵ.

Since

$$\sup_{y_1,y_2\in B_\epsilon(x_i)} \|\psi(y_1)-\psi(y_2)\|_{\mathcal{Y}_2} \le C \sup_{y_1,y_2\in B_\epsilon(x_i)} \|y_1-y_2\|_{\mathcal{Y}_1}^\rho,$$

there exists $x_i' \in \mathcal{Y}_2$ such that

$$\psi(B_\epsilon(x_i)) \subset B_{C\epsilon^\rho}(x_i') \subset \mathcal{Y}_2.$$

It follows from (2.10) that

$$\psi(\mathcal{U}) \subset \bigcup_{j=1}^{N_\epsilon(\mathcal{U})} \psi(B_\epsilon(x_i)) \subset \bigcup_{j=1}^{N_\epsilon(\mathcal{U})} B_{C\epsilon^\rho}(x_i').$$

Then $\psi(\mathcal{U})$ can be covered by at least $N_\epsilon(\mathcal{U})$ balls with the diameter $C\epsilon^\rho$, which implies $N_{C\epsilon^\rho}(\psi(\mathcal{U})) \le N_\epsilon(\mathcal{U})$. Therefore, we arrive at

$$\begin{aligned} d_F(\psi(\mathcal{U})) &= \limsup_{\epsilon\to 0} \frac{\log N_{C\epsilon^\rho}(\psi(\mathcal{U}))}{\log(1/C\epsilon^\rho)} \\ &\le \limsup_{\epsilon\to 0} \frac{\log N_\epsilon(\mathcal{U})}{\rho\log(1/\epsilon)} \\ &= \frac{1}{\rho} d_F(\mathcal{U}), \end{aligned}$$

which completes the proof of the claim.

3. Local Solution.

The local solution will be constructed by the semigroup method (Proposition 2.1) and the truncation method. We consider an auxiliary equation of (OR):

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_u \Delta u + \dfrac{1}{\varepsilon}\{u(1-u)-(fv+\phi_u)(u-q)g(\mathrm{Re}\,u)\} \quad \text{in} \quad \Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial t} = D_v \Delta v + u - v \quad \text{in} \quad \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x),\ v(x,0)=v_0(x) \quad \text{in} \quad \Omega, \end{cases} \tag{$\widetilde{\mathrm{OR}}$}$$

where $g(\xi)$ is an extension function of $\frac{1}{\xi+q}$ such that

$$g(\xi) = \begin{cases} \frac{1}{\xi+q}, & \xi \in [\frac{q}{2}, \infty), \\ \text{smooth extension with } |g(\xi)| \le \frac{1}{q-\xi}, \ |g'(\xi)| + |g''(\xi)| \le C, & \xi \in (-\infty, \frac{q}{2}). \end{cases}$$

We remark that in this system u and v are extended to the complex valued functions in order to apply Proposition 2.1. In view of the proposition by precise setting of H, $\mathcal{D}(A^\alpha)$ and $\mathcal{D}(A^\eta)$ we have an existence theorem of local solution.

Theorem 3.1. *Assume that $(u_0, v_0) \in \mathbb{H}^2_N(\Omega)$ such that $\|(u_0, v_0)\|_{\mathbb{H}^2} \le r$ and r is a positive constant. Then, there exists a unique local solution (u, v) to $(\widetilde{\mathrm{OR}})$ such that*

$$(u, v) \in \mathcal{C}([0, T_r]; \mathbb{H}^2(\Omega)) \cap \mathcal{C}^1((0, T_r]; \mathbb{H}^1(\Omega)) \cap \mathcal{C}((0, T_r]; \mathbb{H}^3_N(\Omega)).$$

Moreover, the following estimates hold:

$$\sqrt{t}\|(u, v)\|_{\mathbb{H}^3} + \|(u, v)\|_{\mathbb{H}^2} \le C_r, \quad 0 < t \le T_r.$$

Proof. The system $(\widetilde{\mathrm{OR}})$ is formed as an abstract semilinear evolution equation (2.6) for $U = (u, v)$, $U_0 = (u_0, v_0)$, $A = \begin{pmatrix} -D_u\Delta + 1 & 0 \\ 0 & -D_v\Delta + 1 \end{pmatrix}$ and

$$F(U) = \begin{pmatrix} u + \dfrac{1}{\varepsilon}\{u(1-u) - (fv + \phi_u)(u-q)g(\mathrm{Re}\, u)\} \\ u \end{pmatrix}.$$

Let us define $H = \mathbb{H}^1(\Omega)$, $\mathcal{D}(A) = \mathbb{H}^3_N(\Omega)$ and $\alpha = \eta = 1/2$. In the setting, (2.7) and (2.8) are clearly satisfied. We shall check the condition (2.9) in Proposition 2.1. We write $\tilde{U} = (\tilde{u}, \tilde{v})$ and $\tilde{U}_0 = (\tilde{u}_0, \tilde{v}_0)$. Let us consider first the Lipschitz continuity of $v(u-q)g(\mathrm{Re}\, u)$ in $H^1(\Omega)$. From the difinition of g we have $\sup_{\xi\in\mathbb{R}} |g(\xi)| \le 2/q$. It is easy to see from (2.5) that

$$\begin{aligned} &\|v(u-q)g(\mathrm{Re}\, u) - \tilde{v}(\tilde{u}-q)g(\mathrm{Re}\, \tilde{u})\|_{H^1} \\ &\quad \le \|(v-\tilde{v})(u-q)g(\mathrm{Re}\, u)\|_{H^1} + \|\tilde{v}(u-\tilde{u})g(\mathrm{Re}\, u)\|_{H^1} \\ &\qquad + \|\tilde{v}(\tilde{u}-q)\{g(\mathrm{Re}\, u) - g(\mathrm{Re}\, \tilde{u})\}\|_{H^1} \\ &\quad \le C\{(\|u\|_{H^2} + 1)\|v-\tilde{v}\|_{H^2} + \|\tilde{v}\|_{H^2}\|u-\tilde{u}\|_{H^2} \\ &\qquad + (\|\tilde{u}\|_{H^2} + 1)\|\tilde{v}\{g(\mathrm{Re}\, u) - g(\mathrm{Re}\, \tilde{u})\}\|_{H^1}\}. \end{aligned}$$

Here, by noting that

$$g(\mathrm{Re}\, u) - g(\mathrm{Re}\, \tilde{u}) = \int_0^1 g'((1-\theta)(\mathrm{Re}\, \tilde{u}) + \theta(\mathrm{Re}\, u))d\theta\{\mathrm{Re}\,(u-\tilde{u})\},$$

we have

$$\|\tilde{v}\{g(\mathrm{Re}\, u) - g(\mathrm{Re}\, \tilde{u})\}\|_{H^1} \le C\|\tilde{v}(u-\tilde{u})\|_{H^1} \le C\|\tilde{v}\|_{H^2}\|u-\tilde{u}\|_{H^2}.$$

Hence, it is obtained that

$$\begin{aligned} &\|v(u-q)g(\mathrm{Re}\, u) - \tilde{v}(\tilde{u}-q)g(\mathrm{Re}\, \tilde{u})\|_{H^1} \\ &\quad \le C(\|u\|_{H^2} + 1)\|v-\tilde{v}\|_{H^2} + C\|\tilde{v}\|_{H^2}(\|\tilde{u}\|_{H^2} + 1)\|u-\tilde{u}\|_{H^2} \\ &\quad \le C(\|A^\alpha U\|_H + \|A^\alpha \tilde{U}\|_H + 1)(\|A^\eta U\|_H + \|A^\eta \tilde{U}\|_H + 1)\|A^\alpha(U-\tilde{U})\|_H. \end{aligned}$$

For the Lipschitz continuity of $\phi_u(t)(u-q)g(\mathrm{Re}\,u)$ in $H^1(\Omega)$, similarly we have

$$\begin{aligned}
&\|\phi_u(u-q)g(\mathrm{Re}\,u)-\phi_{\tilde u}(\tilde u-q)g(\mathrm{Re}\,\tilde u)\|_{H^1}\\
&\quad\le|\phi_u-\phi_{\tilde u}|\|(u-q)g(\mathrm{Re}\,u)\|_{H^1}+|\phi_{\tilde u}|\|(u-\tilde u)g(\mathrm{Re}\,u)\|_{H^1}\\
&\qquad+|\phi_{\tilde u}|\|(\tilde u-q)\{g(\mathrm{Re}\,u)-g(\mathrm{Re}\,\tilde u)\}\|_{H^1}\\
&\quad\le C(\|u\|_{H^1}+1)(\|u\|_{L^p}^{p-1}+\|\tilde u\|_{L^p}^{p-1})\|u-\tilde u\|_{L^p}+C(a\|\tilde u\|_{L^p}^{p}+|b|)\{\|u-\tilde u\|_{H^2}\\
&\qquad+C(\|\tilde u\|_{H^2}+1)\|u-\tilde u\|_{H^2}\}\\
&\quad\le C(\|u\|_{H^2}^{p+1}+\|\tilde u\|_{H^2}^{p+1}+1)(\|u\|_{H^2}+\|\tilde u\|_{H^2}+1)\|u-\tilde u\|_{H^2}\\
&\quad\le CP(\|A^\alpha U\|_H+\|A^\alpha\tilde U\|_H)(\|A^\eta U\|_H\|A^\eta\tilde U\|_H+1)\|A^\alpha(U-\tilde U)\|_H,
\end{aligned}$$

where $P(r)=1+r^{p+1}$. For the Lipschitz continuity of another term, it is easy to show it. We then obtain

$$\begin{aligned}
&\|F(U)-F(\tilde U)\|_H\\
&\quad\le CP(\|A^\alpha U\|_H+\|A^\alpha\tilde U\|_H)(\|A^\eta U\|_H+\|A^\eta\tilde U\|_H+1)\|A^\alpha(U-\tilde U)\|_H.
\end{aligned}\tag{3.1}$$

Therefore, (2.9) is verified. The proof is completed. □

In the last part of this section we verify the boundedness of solution.

Proposition 3.2. *Let (u,v) be the local solution on $[0,T]$ obtained in Theorem 3.1. Provided that the offset constant b as*

$$b\ge -aq^p|\Omega|,\tag{3.2}$$

where $|\Omega|$ is a measure of Ω, and assume that the initial function $(u_0,v_0)\in\mathbb{H}_N^2(\Omega)$ satisfies $q\le u_0(x)\le 1$ and $0\le v_0(x)\le 1$ in Ω. Then, u and v satisfy that

$$q\le u(x,t)\le 1\quad and\quad 0\le v(x,t)\le 1\quad in\quad \Omega\times[0,T].$$

Proof. First, we note that u and v are real-valued. Indeed, the complex conjugates of u and v also satisfy $(\widetilde{\mathrm{OR}})$. Let us show $u\ge q$. Prepare a $\mathcal{C}^3$-function $J_1(\cdot)$ as follows:

$$J_1(u)=\begin{cases}(u-q)^4 & \text{for } u\le q,\\ 0 & \text{for } u\ge q.\end{cases}$$

This function satisfies that

$$\left\{\begin{aligned}
&0\le J_1'(u)(u-q)\le 4J_1(u),\quad u\in(-\infty,\infty);\\
&J_1'(u)<0,\ u<q;\quad J_1'(u)=0,\ u\ge q;\\
&J_1''(u)\ge 0,\quad u\in(-\infty,\infty).
\end{aligned}\right.\tag{3.3}$$

Then we have

$$\begin{aligned}
\varphi_1'(t)&=-D_u\int_\Omega J_1''(u)\,|\nabla u|^2\,dx\\
&\quad+\frac{1}{\varepsilon}\left\{\int_\Omega J_1'(u)u(1-u)\,dx-\int_\Omega J_1'(u)(fv+\phi_u)(u-q)g(\mathrm{Re}\,u)\,dx\right\}.
\end{aligned}$$

Here,

$$\begin{aligned}\int_\Omega J_1'(u)u(1-u)\,dx &= \left(\int_{u<0} + \int_{0\le u<q} + \int_{u\ge q}\right) J_1'(u)u(1-u)\,dx\\ &\le \int_{u<0} J_1'(u)u(1-u)\,dx\\ &= \int_{u<0} J_1'(u)(u-q)\frac{u(1-u)}{u-q}\,dx\\ &\le \int_{u<0} J_1'(u)(u-q)\{|u|+(1-q)\}\,dx\\ &\le C(\|u\|_{H^2}+1)\int_\Omega J_1(u)\,dx\\ &\le C_u\varphi_1(t).\end{aligned}$$

Meanwhile,

$$\begin{aligned}-\int_\Omega J_1'(u)(fv+\phi_u)(u-q)g(u)\,dx &\le \int_{u<q} J_1'(u)(u-q)|fv+\phi_u||g(u)|\,dx\\ &\le C(\|v\|_{H^2}+|\phi_u|)\int_\Omega J_1(u)\,dx\\ &\le C_{u,v}\varphi_1(t).\end{aligned}$$

Therefore, we have $\varphi_1'(t) \le C_{u,v}\varphi_1(t)$. Noting that $\varphi_1(t) \ge 0$ and $\varphi_1(0) = 0$, it is obtained that $\varphi_1(t) = 0$. This shows $u \ge q$. As for $v \ge 0$, by quite similar technique we obtain it.

Next, we shall show $u \le 1$. Prepare here another C^3-function $J_2(\cdot)$ as follows:

$$J_2(u) = \begin{cases}(u-1)^4 & \text{for } u > 1,\\ 0 & \text{for } u \le 1.\end{cases}$$

This function satisfies that

$$\text{(3.4)}\qquad \begin{cases} J_2(u) > 0, \quad u > 1; \qquad J_2(u) = 0, \quad q \le u \le 1,\\ J_2'(u) \ge 0, \quad u > 1; \qquad J_2'(u) = 0, \quad q \le u \le 1,\\ \qquad J_2''(u) \ge 0, \quad u \ge q.\end{cases}$$

Define

$$\varphi_2(t) = \int_\Omega J_2(u(x,t))dx, \quad 0 \le t \le T.$$

Then, by noting that $\|u\|_{L^p}^p \ge q^p|\Omega|$ we have

$$\begin{aligned}\varphi_2'(t) &= -D_u\int_\Omega J_2''(u)\,|\nabla u|^2\,dx + \frac{1}{\varepsilon}\int_\Omega J_2'(u)\left\{u(1-u)-(fv+\phi_u)\frac{u-q}{u+q}\right\}dx\\ &\le \frac{1}{\varepsilon}\int_{u>1} J_2'(u)\frac{u-q}{u+q}\left\{\frac{u(1-u)(u+q)}{u-q} - aq^p|\Omega| - b\right\}dx.\end{aligned}$$

Provided the constant b as (3.2), it is clear that

$$\frac{u(1-u)(u+q)}{u-q} - aq^p|\Omega| - b \le 0 \quad \text{for} \quad u \ge 1.$$

Therefore, we have $\varphi_2'(t) \le 0$; hence $\varphi_2(t) = 0$. This implies $u \le 1$. An argument similar to the above yields $v \le 1$. □

In this section, we constructed the time local solution for the auxiliary equations $(\widetilde{\mathrm{OR}})$, and the solution (u, v) remains to the intervals $[q, 1] \times [0, 1]$. It holds that

$$g(\mathrm{Re}\,\xi) = \frac{1}{\xi + q} \qquad \text{for } \xi \in [q, 1].$$

Hence, the solution of $(\widetilde{\mathrm{OR}})$ coincide with the solution of (OR), which implies the unique local existence of (OR).

4. A Priori Estimates and Global Solution.

In this section we shall first give several apriori estimates.

Lemma 4.1. *Let $(u(t), v(t))$ for $0 \le t \le T$ be the local solution to* (OR) *of Theorem 3.1 of an initial function $(u_0, v_0) \in \mathbb{H}^1(\Omega)$ with $u_0 \in [q, 1]$ and $v_0 \in [0, 1]$. Then, there exists a positive constant exponent δ independent of (u, v) such that*

$$\|(u(t), v(t))\|_{\mathbb{H}^1} \le C(e^{-\delta t}\|(u_0, v_0)\|_{\mathbb{H}^1} + 1), \quad 0 \le t \le T. \tag{4.1}$$

Proof. We first show an estimate of $\mathbb{L}^2$-norm of (u, v). Multiply the first equation of (OR) by u and integrate the product in Ω. Then,

$$\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega u^2 dx + \varepsilon D_u \int_\Omega |\nabla u|^2\, dx = \int_\Omega u^2 dx - \int_\Omega u^3 dx - \int_\Omega u(fv + \phi_u)\frac{u - q}{u + q} dx.$$

Here, noting that

$$\int_\Omega u^2 dx - \int_\Omega u^3 dx \le -\int_\Omega u^2 dx + 2|\Omega|,$$

and

$$\begin{aligned} -\int_\Omega u(fv + \phi_u)\frac{u - q}{u + q} dx &= -\int_\Omega u(fv + a\|u(t)\|_{L^p}^p)\frac{u - q}{u + q} dx - b\int_\Omega u\frac{u - q}{u + q} dx \\ &\le |b|\int_\Omega u dx \le \frac{1}{4}\int_\Omega u^2 dx + b^2|\Omega|. \end{aligned}$$

Therefore, we obtain

$$\varepsilon\frac{d}{dt}\|u\|_{L^2}^2 + \frac{3}{2}\|u\|_{L^2}^2 + 2\varepsilon D_u\|\nabla u\|_{L^2}^2 \le 2(b^2 + 2)|\Omega|. \tag{4.2}$$

Meanwhile, multiplying the second equation of (OR) by v and integrating the product in Ω, we have

$$\frac{d}{dt}\|v\|_{L^2}^2 + 2D_v\|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 \le \|u\|_{L^2}^2. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\begin{aligned} \frac{d}{dt}(\varepsilon\|u\|_{L^2}^2 + \|v\|_{L^2}^2) + 2(\varepsilon D_u\|\nabla u\|_{L^2}^2 + D_v\|\nabla v\|_{L^2}^2) + \frac{1}{2}\|u\|_{L^2}^2 + \|v\|_{L^2}^2 \\ \le 2(b^2 + 2)|\Omega|. \end{aligned} \tag{4.4}$$

By choosing $\tilde{\delta} = \min\{1/2\varepsilon, 1\}$ we have

$$\varepsilon\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 \le e^{-\tilde{\delta}t}(\varepsilon\|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2) + \frac{2(b^2+2)|\Omega|}{\delta}, \quad 0 \le t < T. \tag{4.5}$$

Next we show an estimate of $\mathbb{H}^1$-norm of (u, v). Multiply the first equation of (OR) by Δu and integrate the product in Ω. Then,

$$\begin{aligned}\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \varepsilon D_u \int_\Omega |\Delta u|^2\,dx + 2\int_\Omega u\,|\nabla u|^2\,dx + \int_\Omega (fv + a\|u\|_{L^p}^p)\frac{2q}{(u+q)^2}\,|\nabla u|^2\,dx \\ = \int_\Omega |\nabla u|^2\,dx - f\int_\Omega \frac{u-q}{u+q}\nabla u\cdot\nabla v dx - b\int_\Omega \frac{2q}{(u+q)^2}\,|\nabla u|^2\,dx.\end{aligned}$$

It then follows that

$$\begin{aligned}\varepsilon\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + 2\varepsilon D_u\int_\Omega |\Delta u|^2\,dx + 4q\int_\Omega |\nabla u|^2\,dx \\ \le \left(3 + \frac{|b|}{q}\right)\int_\Omega |\nabla u|^2\,dx + f^2\int_\Omega |\nabla v|^2\,dx.\end{aligned}$$

By noting that $(3 + \frac{|b|}{q})\int_\Omega |\nabla u|^2\,dx \le \varepsilon D_u \int_\Omega |\Delta u|^2\,dx + \frac{(3+\frac{|b|}{q})^2}{4\varepsilon D_u}\int_\Omega u^2 dx$, we have

$$\varepsilon\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \varepsilon D_u\|\Delta u\|_{L^2}^2 + 4q\|\nabla u\|_{L^2}^2 \le \frac{(3+\frac{|b|}{q})^2}{4\varepsilon D_u}\|u\|_{L^2}^2 + f^2\|\nabla v\|_{L^2}^2.$$

Meanwhile, multiplying the second equation of (OR) by Δv and integrating the product in Ω, we get

$$\frac{d}{dt}\|\nabla v\|_{L^2}^2 + D_v\|\Delta v\|_{L^2}^2 + 2\|\nabla v\|_{L^2}^2 \le \frac{1}{D_u}\|u\|_{L^2}^2.$$

Combining these inequalities yields

$$\begin{aligned}\frac{d}{dt}(\varepsilon\|\nabla u\|_{L^2}^2 + f^2\|\nabla v\|_{L^2}^2) + \varepsilon D_u\|\Delta u\|_{L^2}^2 + f^2 D_v\|\Delta v\|_{L^2}^2 \\ + 4q\|\nabla u\|_{L^2}^2 + f^2\|\nabla v\|_{L^2}^2 \le C\|u\|_{L^2}^2.\end{aligned} \tag{4.6}$$

Then in view of (4.5) it holds that

$$\frac{d}{dt}(\varepsilon\|\nabla u\|_{L^2}^2 + f^2\|\nabla v\|_{L^2}^2) + \delta(\varepsilon\|\nabla u\|_{L^2}^2 + f^2\|\nabla v\|_{L^2}^2) \le e^{-\tilde{\delta}t}(\varepsilon\|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2) + C.$$

Choosing δ as $0 < \delta < \min\{4q/\varepsilon, 1, \tilde{\delta}\}$, we conclude that

$$\begin{aligned}&\varepsilon\|\nabla u(t)\|_{L^2}^2 + f^2\|\nabla v(t)\|_{L^2}^2 \\ &\le e^{-\delta t}(\varepsilon\|\nabla u_0\|_{L^2}^2 + f^2\|\nabla v_0\|_{L^2}^2) + Ce^{-\delta t}(\varepsilon\|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2) + C, \quad 0 \le t < T.\end{aligned} \tag{4.7}$$

By combining (4.5) and (4.7) it indicates (4.1). □

In the case where $U_0 \in \mathbb{H}_N^2(\Omega)$, we have another a priori estimate of $\mathbb{H}^2$-norm.

Theorem 4.2. *Let (u, v), $0 \le t \le T$, be the local solution to (OR) of Theorem 3.1 of an initial function $(u_0, v_0) \in \mathbb{H}_N^2(\Omega)$ with $u_0 \in [q, 1]$ and $v_0 \in [0, 1]$. Then, there exist a positive constant exponent δ and an increasing continuous function $P(\cdot)$ independent of (u, v) such that*

$$\|(u(t), v(t))\|_{\mathbb{H}^2} \le Ce^{-\delta t}\|\Delta(u_0, v_0)\|_{\mathbb{L}^2} + P(\|(u_0, v_0)\|_{\mathbb{H}^1}), \quad 0 \le t \le T. \tag{4.8}$$

Proof. Operating ∇ to the first equation of (OR), multiplying it by $\nabla \Delta u$ and integrating the product in Ω, we have

$$\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2\,dx + \varepsilon D_u \int_\Omega |\nabla \Delta u|^2\,dx \le \int_\Omega |\nabla\Delta u \nabla u| dx$$
$$+\left(f+\frac{2|\phi_u|}{q}\right)\int_\Omega |\nabla\Delta u\nabla v|dx + 2\int_\Omega |u\nabla\Delta u\nabla u|dx + \frac{2f}{q}\int_\Omega |v\nabla\Delta u\nabla v|dx.$$

All integrals of the right-hand side are estimated from above by $\zeta\int_\Omega |\nabla\Delta u|^2 dx + p_\zeta(\|u\|_{H^1}+\|v\|_{H^1})$ with ζ arbitrary. Indeed, in view of

$$\int_\Omega u^2|\nabla u|^2 dx \le \|u\|_{L^4}^2\|\nabla u\|_{L^4}^2 \le C\|u\|_{L^2}\|u\|_{H^1}^2\|u\|_{H^2}$$
$$\le C(\|u\|_{H^1}^6+1) + C\left(-\int_\Omega \nabla\Delta u\nabla u dx\right)$$
$$\le \zeta^2\int_\Omega |\nabla\Delta u|^2 dx + C_\zeta(\|u\|_{H^1}^6+1)$$

it is true that

$$\int_\Omega |u\nabla\Delta u\nabla u|dx \le \frac{\zeta}{2}\int_\Omega |\nabla\Delta u|^2dx + \frac{1}{2\zeta}\int_\Omega u^2|\nabla u|^2 dx$$
$$\le \zeta\int_\Omega |\nabla\Delta u|^2 dx + C_\zeta(\|u\|_{H^1}^6+1).$$

Similarly, from

$$\int_\Omega v^2|\nabla u|^2 dx \le \frac{\zeta^2 q^2}{4}\int_\Omega |\nabla\Delta u|^2 dx + P(\|u\|_{H^1}+\|v\|_{H^1}),$$

we obtain

$$\int_\Omega |v\nabla\Delta u\nabla u|dx \le \frac{\zeta}{2}\int_\Omega |\nabla\Delta u|^2 dx + \frac{2}{\zeta q^2}\int_\Omega v^2|\nabla u|^2 dx$$
$$\le \zeta\int_\Omega |\nabla\Delta u|^2 dx + P(\|u\|_{H^1}+\|v\|_{H^1}).$$

The other integrals are similarly estimated. By noting that

$$\int_\Omega |\Delta u|^2\,dx \le \frac{D_u}{2}\int_\Omega |\nabla\Delta u|^2\,dx + \frac{1}{2D_u}\int_\Omega |\nabla u|^2\,dx,$$

hence, we have

$$\varepsilon\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \frac{\varepsilon D_u}{2}\|\nabla\Delta u\|_{L^2}^2 + \varepsilon\|\Delta u\|_{L^2}^2 \le P(\|u\|_{H^1}+\|v\|_{H^1})$$
$$\le P(\|u_0\|_{H^1}+\|v_0\|_{H^1}),$$

with the aid of (4.1). Solving this,

$$\|\Delta u(t)\|_{L^2}^2 \le e^{-t}\|\Delta u_0\|_{L^2}^2 + P(\|u_0\|_{H^1}+\|v_0\|_{H^1}), \quad 0\le t<T. \tag{4.9}$$

Meanwhile, operate ∇ to the second equation of (OR) and multiply it by $\nabla\Delta v$, and integrate the product in Ω. Then,

$$\frac{d}{dt}\|\Delta v\|_{L^2}^2 + D_v\|\nabla\Delta v\|_{L^2}^2 + 2\|\Delta v\|_{L^2}^2 \le \frac{1}{D_u}\|\nabla u\|_{L^2}^2.$$

Solving this,

$$\|\Delta v(t)\|_{L^2}^2 \leq e^{-2t}\|\Delta v_0\|_{L^2}^2 + C\{e^{-\delta t}(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2) + 1\}, \quad 0 \leq t < T. \tag{4.10}$$

Combining (4.1), (4.9) and (4.10), we have (4.8). □

By the estimate (4.1) we have a global existence of solutions.

Theorem 4.3. *Let* $(u_0, v_0) \in \mathbb{H}_N^2(\Omega)$ *with* $u_0 \in [q, 1]$ *and* $v_0 \in [0, 1]$. *Then* (OR) *admits a unique global solution* (u, v) *such that*

$$(u, v) \in \mathcal{C}([0, \infty); \mathbb{H}_N^2(\Omega)) \cap \mathcal{C}^1((0, \infty); \mathbb{H}^1(\Omega)) \cap \mathcal{C}((0, \infty); \mathbb{H}_N^3(\Omega)).$$

In addition, u *and* v *satisfy* $u \in [q, 1]$ *and* $v \in [0, 1]$.

Proof. Theorem 3.1 admits the local solution U such that

$$(u, v) \in \mathcal{C}([0, T_r]; \mathbb{H}_N^2(\Omega)) \cap \mathcal{C}^1((0, T_r]; \mathbb{H}^1(\Omega)) \cap \mathcal{C}((0, T_r]; \mathbb{H}_N^3(\Omega)),$$

where T_r is determined by $r = \|(u_0, v_0)\|_{\mathbb{H}^2}$. Then, by Theorem 4.2 we have $\|(u(t), v(t))\|_{\mathbb{H}^2} \leq C(r + 1)$ $(= r')$, $0 \leq t \leq T_r$. Consider now (OR) with an initial function $(u(T_r), v(T_r))$. Theorem 3.1 admits an extension of solution beyond T_r to $T_r + T_{r'}$. By the uniqueness of solution this shows that $\|(u(t), v(t))\|_{\mathbb{H}^2} \leq r'$, $0 \leq t \leq T_r + T_{r'}$. This allows the solution to exist until $T_r + 2T_{r'}$. Repeating this argument, we obtain the global existence of solution. □

5. Exponential Attractor.

The initial value (u_0, v_0) belongs to K, where K is a set of initial values

$$K = \left\{(u_0, v_0) \in \mathbb{H}_N^2; u_0 \in [q, 1],\ v_0 \in [0, 1]\right\}.$$

Theorem 4.3 shows that (5.1) is well-posed in K. Therefore, a continuous semigroup $\{S(t) : (u_0, v_0) \in K \mapsto (u(t), v(t))\}_{t \geq 0}$ is generated by (5.1), and a dynamical system $(\{S(t)\}_{t \geq 0}, K)$ is defined.

For the dynamical system $(\{S(t)\}_{t \geq 0}, K)$ the existence of absorbing set is shown.

Theorem 5.1. *Let* B *be an arbitrary bounded set in* K. *Then, there exist a time* t_B *and a universal constant* R *for* B *such that*

$$\sup_{(u_0, v_0) \in B} \|S(t)(u_0, v_0)\|_{\mathbb{H}^2} \leq R \quad \textit{for all} \quad t \geq t_B.$$

Therefore, a compact set $\mathcal{B} = \{(u, v) \in \mathbb{H}_N^2;\ \|(u, v)\|_{\mathbb{H}^2} \leq R\}$ *in* $\mathbb{H}^1$ *is an absorbing set for the dynamical system* $(\{S(t)\}_{t \geq 0}, K)$.

Proof. Let $(u_0, v_0) \in B$ and $(u(t), v(t)) = S(t)(u_0, v_0)$. By (4.8) it holds that

$$\|(u(t), v(t))\|_{\mathbb{H}^2} \leq Ce^{-\delta t}\|\Delta(u(s), v(s))\|_{\mathbb{L}^2} + P(\|(u(s), v(s))\|_{\mathbb{H}^1}), \quad t \geq s \geq 0.$$

But, from (4.1) we have

$$\|(u(t), v(t))\|_{\mathbb{H}^2} \leq Ce^{-\delta t}\|\Delta(u(s), v(s))\|_{\mathbb{L}^2} + P(e^{-\delta s}\|(u_0, v_0)\|_{\mathbb{H}^1}), \quad t \geq s \geq 0.$$

Choose $s_0 = \max\{\delta^{-1} \log \|(u_0, v_0)\|_{\mathbb{H}^1}, 0\}$. Then, we obtain

$$\|(u(t), v(t))\|_{\mathbb{H}^2} \leq C(e^{-\delta t}\|\Delta(u(s_0), v(s_0))\|_{\mathbb{L}^2} + 1), \quad t \geq s_0 \geq 0.$$

Choose also $t_0 = \max\{\delta^{-1}\log\|\Delta(u(s_0),v(s_0))\|_{\mathbb{H}^1}, s_0\}$. Then, putting $R = 2C$ and $t_B = t_0$, we obtain

$$\|(u(t),v(t))\|_{\mathbb{H}^2} \le R, \quad t \ge t_B.$$

Thus, we have proved the theorem. □

Hence, by virtue of [7, Chap. I, Theorem 1.1], the ω-limit set $\mathcal{A} = \omega(\mathcal{B})$ is a global attractor for $(\{S(t)\}_{t\ge0}, K)$. Set now

$$\mathcal{X} = \overline{\bigcup_{t\ge t_{\mathcal{B}}} S(t)\mathcal{B}}^{\mathbb{H}^1},$$

where $t_{\mathcal{B}}$ is a number satisfying $S(t)\mathcal{B} \subset \mathcal{B}$ for $t \ge t_{\mathcal{B}}$. The set $\mathcal{X}$ is a compact subset of $\mathbb{H}^1$ with $\mathcal{A} \subset \mathcal{X} \subset \mathcal{B}$, and is an absorbing and positively invariant set for $(\{S(t)\}_{t\ge0}, K)$. Therefore to investigate the large time behavior of solution it suffices to consider a dynamical system $(\{S(t)\}_{t\ge0}, \mathcal{X})$.

By using Proposition 2.2, we obtain the following theorem:

Theorem 5.2. *The dynamical system $(\{S(t)\}_{t\ge0}, \mathcal{X})$ admits an exponential attractor $\mathcal{M}$.*

Proof. Let $H = \mathbb{H}^1(\Omega)$. We consider a semilinear equation in H:

$$(5.1)\qquad \begin{cases} \dfrac{dU}{dt} + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0, \end{cases}$$

where $U = (u,v)$. The linear operator A is defined as $A = \begin{pmatrix} -D_u\Delta + 1 & 0 \\ 0 & -D_v\Delta + 1 \end{pmatrix}$ with the domain $\mathcal{D}(A) = \mathbb{H}^3_N(\Omega)$, and $F(U)$ is a nonlinear operator from $\mathcal{D}(A)$ to H such that

$$F(U) = \begin{pmatrix} u + \dfrac{1}{\varepsilon}\left\{u(1-u) - (fv + \phi_u)\dfrac{u-q}{u+q}\right\} \\ u \end{pmatrix}.$$

Let us check the conditions (F) and (G) in Proposition 2.2. Let $U_0 \in \mathcal{X}$. Since $\|A^{\frac12}U_0\|_H \le R$, we have $\|A^{\frac12}U(t)\|_H \le R$ for every $t \ge 0$. In view of (3.1) we obtain

$$\begin{aligned} \|F(U) - F(\tilde U)\|_H &\le CP(\|A^{\frac12}U\|_H + \|A^{\frac12}\tilde U\|_H)(\|A^{\frac12}U\|_H + \|A^{\frac12}\tilde U\|_H + 1)\|A^{\frac12}(U - \tilde U)\|_H \\ &\le C_R\|A^{\frac12}(U - \tilde U)\|_H, \quad U, \tilde U \in \mathcal{X}. \end{aligned}$$

Therefore, the condition (F) is fulfilled.

Next, we check the condition (G). Fix $T > 0$ arbitrarily. We note that

$$\|G(t,U_0) - G(s,\tilde U_0)\|_H \le \|S(t)U_0 - S(t)\tilde U_0\|_H + \|S(t)\tilde U_0 - S(s)\tilde U_0\|_H.$$

For $U_0, \tilde U_0 \in \mathcal{X}$, let $W(t) = S(t)U_0 - S(t)\tilde U_0$, $0 \le t \le T$. Obviously $W(t)$ is a solution to the problem

$$(5.2)\qquad \begin{cases} \dfrac{dW}{dt} + AW = F(S(t)U_0) - F(S(t)\tilde U_0), & 0 < t \le T, \\ W(0) = W_0, \end{cases}$$

where $W_0 = U_0 - \tilde U_0$. Multiplying W the equation of (5.2), we have

$$\frac12\frac{d}{dt}\|W\|_H^2 + \|A^{\frac12}W\|_H^2 = (F(S(t)U_0) - F(S(t)\tilde U_0), W)_H \le C_R\|A^{\frac12}W\|_H\|W\|_H.$$

Solving this differential inequality, we have $\|W(t)\|_H \le e^{Ct}\|W_0\|_H \le C_T\|W_0\|_H$, namely,

$$\|S(t)U_0 - S(t)\tilde{U}_0\|_H \le C_T\|U_0 - \tilde{U}_0\|_H.$$

By a quite similar estimate to (3.1) we can show

$$\|F(S(t)\tilde{U}_0)\|_H \le C(\|A^{\frac{1}{2}}S(t)\tilde{U}_0\|_H + 1)^2 \le C_R, \quad t \ge 0.$$

Therefore, we observe that for $0 \le s \le t \le T$

$$\begin{aligned}
\|S(t)\tilde{U}_0 - S(s)\tilde{U}_0\|_H &\le \int_s^t \left\| \frac{dS}{dt}(\tau)\tilde{U}_0 \right\|_H d\tau \\
&\le \int_s^t \|AS(\tau)\tilde{U}_0\|_H d\tau + \int_s^t \|F(S(\tau)\tilde{U}_0)\|_H d\tau \\
&\le \sup_{\tau\in[s,t]} \left(\sqrt{\tau}\|AS(\tau)\tilde{U}_0\|_H\right) \int_s^t \frac{1}{\sqrt{\tau}} d\tau + \sup_{\tau\in[s,t]} \|F(S(\tau)\tilde{U}_0)\|_H \int_s^t d\tau \\
&\le C_R\sqrt{t-s} + C_R(t-s) \\
&\le C_{R,T}\sqrt{t-s}.
\end{aligned}$$

Thus the condition (G) is also fulfilled. □

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References

[1] A. Eden, C. Foias, B. Nicolaenko, and R. Temam, *Exponential attractors for dissipative evolution equations*, Research in Applied Mathematics, vol. 37, John-Wiley and Sons, New York (1994), 25–33.

[2] R. J. Field and R. M. Noyes, *Oscillations in chemical systems. IV. Limit cycle behavior in a model of a real chemical reaction*, J. Chem. Phys. **60** (1974), 1877–1884.

[3] H.-J. Krug., L. Pohlmann, and L. Kuhnert, *Analysis of the modified complete Oregonator accounting for oxygen sensitivity and photosensitivity of Belousov-Zhabotinskii systems*, J. Phys. Chem. **94** (1990), 4862–4866.

[4] E. Mihaliuk, T. Sakurai, F. Chirila, and K. Showalter, *Feedback Stabilization of Unstable Propagating Waves*, Physical Review E **65** (2002), 065602.

[5] I. Sendina Nadal, E. Mihaliuk, J. Wang, V. Pŕez-Munuzuri, and K. Showalter, *Wave Propagation in Subexcitable Media with Periodically Modulated Excitability*, Phys. Rev. Lett. **86** (2001), 1646–1649.

[6] K. Osaki and A. Yagi, *Global Existence for a Chemotaxis-Growth System in* $\mathbb{R}^2$, Adv. Math. Sci. Appl. **12(2)** (2002), 587–606.

[7] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics, second edition*, Applied mathematical sciences, vol. 68, Springer-Verlag, New York (1997), 15–42.

[8] J. J. Tyson and P. C. Fife, *Target patterns in a realistic model of the Belousov-Zhabotinskii reaction*, J. Chem. Phys **73** (1980), 2224–2237.

[9] E. Zeidler, *Nonlinear Functional Analysis and its applications II/A*, Springer-Verlag (1990), 168–173.