Asymptotic Behavior of Reaction-Diffusion-Advection Systems

Koichi Osaki*

Abstract. Two reaction-diffusion-advection systems: Mikhailov-Hildebrand-Ertl model [1] and Mimura-Tsujikawa model [2] are considered. As an example of reaction-diffusion-advection systems, Mikhailov-Hildebrand-Ertl model in $\mathbb{R}^2$ is adopted, and then the method of showing the global existence: semigroup method and a priori estimate is introduced. As another topic of the asymptotic behavior of reaction-diffusion-advection systems, the collapse of solution is treated. For Mimura-Tsujikawa model the possibility of occurrence of collapse due to the relation cross-diffusion and growth orders is discussed.

Key words: reaction-diffusion system, advection, chemotaxis, global existence, blow up.

1. Introduction.

In this paper we consider the following type of reaction-diffusion-advection systems:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla\{V(u)\nabla v\} + f(u), & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \alpha \Delta v + g(u,v) & \text{in} \quad \Omega \times (0,\infty), \end{cases}$$

where Ω is a domain in $\mathbb{R}^2$, and χ and α are positive constants. The function $V(u)$ of the advection term $-\chi\nabla\{V(u)\nabla v\}$ denotes a cross-diffusion effect of u by the gradient of v, and $f(u)$ and $g(u,v)$ denote the reactions between components u and v or simply growth of u and v. We shall consider the functions as:

$$(1.2)\qquad V(u) = u(1-u),\ f(u) = 1-u,\ g(u,v) = v(u+v-1)(1-v)$$

or

$$(1.3)\qquad V(u) = u,\ f(u) = 1-u,\ g(u,v) = \beta u - \gamma v,$$

where β and γ are positive constants. The system (1.1) with (1.2) is equivalent to Mikhailov-Hildebrand-Ertl model [1] in terms of the method of showing the global existence essentially. The system (1.1) with (1.3) is Mimura-Tsujikawa model [2] having a linear decay growth term.

The objective of this paper is to introduce the treatment of showing the global existence of reaction-diffusion-advection systems: semigroup method and a priori estimate by adopting Mikhailov-Hildebrand-Ertl model as one of examples of the systems. As a symbolic asymptotic behavior of reaction-diffusion-advection systems, we consider also collapse of solution by adopting Mimura-Tsujikawa model. Collapse means here that the function $u(x,t)$ has a delta function singularity in a finite time because of the effect of negative diffusion due to the advection term. For no growth case, that is, Keller-Segel model: $V(u) = u$, $f(u) \equiv 0$, $g(u,v) = \beta u - \gamma v$, it is well-known that collapse occurs for sufficiently large $\chi > 0$ (results on the local and global existence and collapse for Keller-Segel model e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11]). On the other hand, for a quadratic decay growth

(２００６年１１月２日 受理)

*Department of Business Administration, Ube National College of Technology.

case [12]: $V(u) = u$, $f(u) = u(1-u)$, $g(u,v) = \beta u - \gamma v$ or a cross-diffusion with prevention of overcrowding case [13]: $V(u) = u(1-u)$, $f(u) = 0$, $g(u,v) = \beta u - \gamma v$ the global existence were assured for any $\chi > 0$. Hence, to investigate the effect of cross-diffusion and growth orders for the collapse may be important to know the asymptotic behavior of reaction-diffusion-advection systems. In this paper, by our results of Mikhailov-Hildebrand-Ertl model we observe that the case: $V(u) = u(1-u)$, $f(u) = 1-u$, $g(u,v) = \beta u - \gamma v$ assure the global existence for any $\chi > 0$. We treat also the Mimura-Tsujikawa model with a linear decay growth case (1.3), and discuss on the possibility of occurrence of the collapse for sufficiently large $\chi > 0$.

The organization of this paper is as follows. In Section 2 we shall treat Mikhailov-Hildebrand-Ertl model in the spatial domain $\mathbb{R}^2$ and show the global existence of solutions. Section 3 is devoted to the discussion on collapse of Mimura-Tsujikawa model.

2. Global Solution to Mikhailov-Hildebrand-Ertl Model.

Let us consider a following reaction-diffusion-advection system which was proposed by Hildebrand, Kuperman, Wio, Mikhailov and Ertl [1] (see also [14]):

$$(2.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + b\nabla\{u(1-u)\nabla\chi(v)\} \\ \qquad\qquad -ce^{k\chi(v)}u - du + f(1-u) \quad \text{in} \quad \mathbb{R}^2\times(0,\infty), \\ \dfrac{\partial v}{\partial t} = g\Delta v + hv(u+v-1)(1-v) \quad \text{in} \quad \mathbb{R}^2\times(0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \quad \text{in} \quad \mathbb{R}^2. \end{cases}$$

Here, u and v are unknown functions with $0 \le u \le 1$ and $0 \le v \le 1$. The coefficients a, b, c, d, f, g, h and k are positive constants, and $\chi(v)$ is a real-valued smooth function on $v \in [0,1]$ with $\chi'(v) \le 0$ (in [1], $\chi(v) = \frac{1}{3}v^3 - \frac{1}{2}v^2$ is introduced).

The system (2.1) is a model for a nonequilibrium self-organization process in surface chemical reaction of microreactors with submicrometer and nanometer sizes. Typical example of the reaction is the oxidation of CO on Pt(110) surface (cf. [15]). Then, the functions u and v denote the adsorbate coverage of CO and a continuous order parameter of the surface structural state of Pt(110), respectively. The advection term $b\nabla\{u(1-u)\nabla\chi(v)\}$ of the first equation shows that CO molecules flow on the surface by the gradient of local potential $\chi(v)$ with a rate $1-u$. The reaction term $hv(u+v-1)(1-v)$ of the second equation indicates that the system has two stable uniform states $v = 0, 1$ and an unstable uniform state $v = 1-u$.

In [16], Tsujikawa and Yagi treated the system in a bounded domain with $\mathcal{C}^3$ boundary, imposed Neumann boundary conditions, and then prove the existence of global solutions and an exponential attractor (cf. [17] for periodic boundary conditions). Exponential attractor is a compact positively invariant set with the finite fractal dimension in the (infinite dimensioal) phase space, which includes the global attractor, and attracts every trajectory in an exponential rate. (As for precise definition and examples of exponential attractor, see Eden, Foias, Nicolaenko and Temam [18]). The system shows various spatial-temporal patterns [15] (cf. numerical results [19]), so, we may consider in terms of the existence of exponential attractor that such patterns are phenomenon of finite degree of freedom even if they seem to be complicated.

In [1, 14, 20] the existence and stability of stationary spots and traveling front solutions to (2.1) are discussed, and also the interface equation is introduced in the domain $\mathbb{R}^2$. So, we should consider the case $\mathbb{R}^2$ and show the global existence of (2.1) (exponential attractor does not exist in a usual function space such as $L^2(\mathbb{R}^2)$, in fact, a norm of some traveling solution may diverge to infinity). Since $\chi'(v) \le 0$, there exist only three stationary solutions such that stable uniform states:

$$(\text{S})\qquad (\tilde{u}, \tilde{v}) = (\tilde{u}_0, 0),\ (\tilde{u}_1, 1),\ \tilde{u}_i = \frac{f}{ce^{k\chi(i)} + d + f},\ i = 0, 1;$$

and an unstable uniform state:

(US) $$(\tilde{u},\tilde{v}) = (\tilde{u}_*,\tilde{v}_*),\ \tilde{u}_* = \frac{f}{ce^{k\chi(1-\tilde{u}_*)}+d+f},\ \tilde{v}_* = 1-\tilde{u}_*.$$

The domain is unbounded, it then may be natural to impose on (2.1) a boundary condition of $\lim_{|x|\to\infty}(u,v) = (\tilde{u},\tilde{v})$. Then, by changing u and v as

(2.2) $$u = \tilde{u} + u_p \quad\text{and}\quad v = \tilde{v} + v_p$$

an evolution equation of perturbation (u_p, v_p) around $(\tilde{u},\tilde{v})$ is derived:

(P) $$\begin{cases} \dfrac{\partial u_p}{\partial t} = a\Delta u_p + b\nabla\{u(1-u)\nabla\chi(v)\} - (ce^{k\chi(v)}+d+f)u_p \\ \qquad\qquad +c\tilde{u}(e^{k\chi(\tilde{v})} - e^{k\chi(v)}) \quad \text{in} \quad \mathbb{R}^2\times(0,\infty), \\ \dfrac{\partial v_p}{\partial t} = g\Delta v_p + hv(u+v-1)(1-v) \quad \text{in} \quad \mathbb{R}^2\times(0,\infty), \\ u_p(x,0) = u_{p,0}(x),\ v_p(x,0) = v_{p,0}(x) \quad \text{in} \quad \mathbb{R}^2, \end{cases}$$

where u, v, $\tilde{u}$ and $\tilde{v}$ are defined in (S) or (US) and (2.2).

As for the case (S), the author, Takei and Tsujikawa show the global existence [21]. In this paper, we treat the case (US). Then, it is obtained that

Theorem 2.1. *Let* $(u_{p,0}, v_{p,0}) \in H^1(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ *with the initial condition:*

(B_0) $$0 \le \tilde{u} + u_{p,0}(x) \le 1, \quad 0 \le \tilde{v} + v_{p,0}(x) \le 1 \quad \text{in} \quad \mathbb{R}^2.$$

Then, there exists a unique global solution (u_p, v_p) *to* (P) *which is an evolution equation of around the unstable uniform state* (US) *such that*

$$(u_p,v_p) \in \mathcal{C}([0,\infty);\, H^1(\mathbb{R}^2)\times H^2(\mathbb{R}^2)) \cap \mathcal{C}^1((0,\infty);\, H^1(\mathbb{R}^2)\times H^2(\mathbb{R}^2)) \\ \cap\, \mathcal{C}((0,\infty);\, H^3(\mathbb{R}^2)\times H^4(\mathbb{R}^2)).$$

In addition, the global solution satisfies on $\mathbb{R}^2\times[0,\infty)$

(B) $$0 \le \tilde{u} + u_p(x,t) \le 1, \quad 0 \le \tilde{v} + v_p(x,t) \le 1.$$

Local Solutions.

We shall show the local existence of solutions to (P). We set (P) to an abstract evolution equation in a function space and by using semigroup method obtain the local existence.

Consider an initial value problem of a semilinear abstract evolution equation:

(2.3) $$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t \le T, \\ U(0) = U_0 \end{cases}$$

in a Banach space X. The function U is an unknown function, and A is a closed linear operator in X which satisfies the condition

(2.4) $$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda|+1}, \quad \lambda \notin \Sigma,$$

with $\Sigma = \{\lambda \in \mathbb{C};\, |\arg\lambda| \le \phi\}$, $0 \le \phi < \frac{\pi}{2}$, and $M > 0$ is a constant. The initial value U_0 is in $\mathcal{D}(A^\alpha)$ with the estimate

(2.5) $$\|A^\alpha U_0\| \le R,$$

where $\alpha \in [0,1)$ is some exponent and $R > 0$ is a constant. These assumptions allow to generate an analytic semigroup e^{-tA} on X and define a function of integral form

$$\{\Phi(U)\}(t) = e^{-tA}U_0 + \int_0^t e^{-(t-s)A}F(U(s))ds.$$

If some Lipschitz condition for the nonlinear function F is satisfied, then it is shown that the function $\Phi(U)$ has a unique fixed point, hence, we obtain the local existence (e.g. [22, 23, 24, 25, 26]). But, for our reaction-diffusion-advection system the following Lipschitz condition of [27] may be easily verified: the function $F : \mathcal{D}(A^\eta) \to X$ is a given Lipschitz continuous function satisfying

$$\begin{aligned} \|F(U) - F(V)\|_X \le\ & p(\|A^\alpha U\|_X + \|A^\alpha V\|_X) \\ & \times\{\|A^\eta(U-V)\|_X + (\|A^\eta U\|_X + \|A^\eta V\|_X)\|A^\alpha(U-V)\|_X\}, \end{aligned} \tag{2.6}$$

where $U,\ V \in \mathcal{D}(A^\eta)$ with some exponent $\eta \in [\alpha, 1)$ and some increasing continuous function $p(\cdot)$.

Proposition 2.2. ([27, Theorem 3.1]) *Under the conditions* (2.4), (2.5) *and* (2.6), *there exists a unique local solution to* (2.3) *in the space*

$$\begin{cases} U \in \mathcal{C}([0,T_R]; \mathcal{D}(A^\alpha)) \cap \mathcal{C}^1((0,T_R]; X) \cap \mathcal{C}((0,T_R]; \mathcal{D}(A)), \\ t^{1-\alpha}U \in \mathcal{B}((0,T_R]; \mathcal{D}(A)), \end{cases}$$

where $T_R > 0$ is determined by R.

Let us set (P) to an abstract evolution equation in X:

$$(\widetilde{\mathrm{P}}) \qquad \begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ U(0) = U_0. \end{cases}$$

Here, $U = \begin{pmatrix} u_p \\ v_p \end{pmatrix}$ and $A = \begin{pmatrix} -a\Delta + d + f & 0 \\ 0 & -g\Delta + 1 \end{pmatrix}$ with the domain $\mathcal{D}(A)$. The initial value $U_0 = \begin{pmatrix} u_{p,0} \\ v_{p,0} \end{pmatrix}$ is in $\mathcal{D}(A^\alpha)$, $\alpha \in [0,1)$, and $F(U)$ is a nonlinear operator from $\mathcal{D}(A^\eta)$ to X, $\eta \in [\alpha, 1)$, such that

$$F(U) = \begin{pmatrix} b\nabla\{u(1-u)\nabla\widetilde{\chi}(\mathrm{Re}\, v)\} - ce^{k\widetilde{\chi}(\mathrm{Re}\, v)}u_p + c\tilde{v}(e^{k\widetilde{\chi}(\tilde{v})} - e^{k\widetilde{\chi}(\mathrm{Re}\, v)}) \\ v_p + hv(u+v-1)(1-v) \end{pmatrix}$$

with $u = \tilde{u} + u_p$ and $v = \tilde{v} + v_p$, where $\widetilde{\chi}(\mathrm{Re}\, v)$ is some smooth extension of $\chi(\mathrm{Re}\, v)$ for $v \in \mathbb{C}$.

The existence theorem of local solutions to $(\widetilde{\mathrm{P}})$ is derived.

Theorem 2.3. *Assume that $(u_{p,0}, v_{p,0}) \in H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ and $\|(u_{p,0}, v_{p,0})\|_{H^1\times H^2} \le R$, where R is some number. Let $X = L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, $\mathcal{D}(A) = H^2(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$. Then, there exists a unique local solution (u_p, v_p) to $(\widetilde{\mathrm{P}})$ such that*

$$(u_p, v_p) \in \mathcal{C}([0,T_R]; H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \cap \mathcal{C}^1((0,T_R]; X) \cap \mathcal{C}((0,T_R]; \mathcal{D}(A))$$

with $T_R > 0$ determined by R.

Sketch of Proof. Let us verify the conditions of Proposition 2.2. The operator A from $\mathcal{D}(A)$ to X satisfies (2.4). We choose $\alpha = 1/2$. This shows that $\mathcal{D}(A^\alpha) = H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$, that is, (2.5) is satisfied. For (2.6) we here mention only the advection term. Choosing $\eta = 3/4$, that is, $\mathcal{D}(A^\eta) = H^{\frac{3}{2}}(\mathbb{R}^2) \times H^{\frac{5}{2}}(\mathbb{R}^2)$, we obtain

$$\begin{aligned}
&\|\nabla [\{u(1-u) - w(1-w)\}\nabla\widetilde{\chi}(\mathrm{Re}\, v)]\|_{L^2} \\
&\qquad \le |1-2\tilde{u}|\|(u_p - w_p)\nabla\widetilde{\chi}(\mathrm{Re}\, v)\|_{H^1} + \|(u_p + w_p)(u_p - w_p)\nabla\widetilde{\chi}(\mathrm{Re}\, v)\|_{H^1} \\
&\qquad \le (\|u_p + w_p\|_{H^{\frac{5}{4}}} + 1)\|u_p - w_p\|_{H^{\frac{5}{4}}}\|\widetilde{\chi}'(\mathrm{Re}\, v)\nabla(\mathrm{Re}\, v_p)\|_{H^1} \\
&\qquad \le (\|u_p + w_p\|_{H^1}^{\frac{1}{2}}\|u_p + w_p\|_{H^{\frac{3}{2}}}^{\frac{1}{2}} + 1)\|u_p - w_p\|_{H^1}^{\frac{1}{2}}\|u_p - w_p\|_{H^{\frac{3}{2}}}^{\frac{1}{2}} p(\|v_p\|_{H^2}) \\
&\qquad \le p(\|v_p\|_{H^2})\{(\|u_p + w_p\|_{H^1} + 1)\|u_p - w_p\|_{H^{\frac{3}{2}}} + \|u_p + w_p\|_{H^{\frac{3}{2}}}\|u_p - w_p\|_{H^1}\} \\
&\qquad \le p(\|u_p\|_{H^1} + \|w_p\|_{H^1} + \|v_p\|_{H^2}) \\
&\qquad\qquad \times \{\|u_p - w_p\|_{H^{\frac{3}{2}}} + (\|u_p\|_{H^{\frac{3}{2}}} + \|w_p\|_{H^{\frac{3}{2}}} + 1)\|u_p - w_p\|_{H^1}\},
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla [u(1-u)\nabla\{\widetilde{\chi}(\mathrm{Re}\, v) - \widetilde{\chi}(\mathrm{Re}\, z)\}]\|_{L^2} \le C(\|u_p\|_{H^{\frac{5}{4}}}^2 + 1) \\
&\qquad \times \left[\|\{\widetilde{\chi}'(\mathrm{Re}\, v) - \widetilde{\chi}'(\mathrm{Re}\, z)\}\nabla(\mathrm{Re}\, v_p)\|_{H^1} + \|\widetilde{\chi}'(\mathrm{Re}\, z)\nabla(v_p - z_p)\|_{H^1}\right] \\
&\qquad \le p(\|u_p\|_{H^1} + \|v_p\|_{H^2} + \|z_p\|_{H^2})(\|u_p\|_{H^{\frac{3}{2}}} + 1)\|v_p - z_p\|_{H^2}.
\end{aligned}$$

The other terms are similarly estimated, the condition (2.6) is verified. Thus, the theorem is proved. □

By considering the higher regularity case for the local solution we obtain the theorem of existence of local solution. But similar arguments as the proof of [21, Theorem 3.4.] derive the following theorem, we then omit the proof and only give the statement:

Theorem 2.4. *Let $(u_{p,0}, v_{p,0}) \in H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ with* (B$_0$) *and $\|(u_{p,0}, v_{p,0})\|_{H^1 \times H^2} \le R$, where R is some number. Then, there exists a unique local solution (u_p, v_p) to* (P) *such that*

$$\begin{aligned}
(u_p, v_p) \in \mathcal{C}([0,T_R];\, H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \cap \mathcal{C}^1((0,T_R];\, H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \\
\cap\, \mathcal{C}((0,T_R];\, H^3(\mathbb{R}^2) \times H^4(\mathbb{R}^2)),
\end{aligned}$$

and satisfying (B) *on $\mathbb{R}^2 \times [0,T_R]$. Here, $T_R > 0$ is determined by R.*

Global Solutions. We shall construct several a priori estimates for the local solutions and then show the global existence of solutions to (P) with (US).

Proposition 2.5. *Let (u_p, v_p) be any local solution to* (P) *with* (US) *which belongs to the function space*

$$\begin{aligned}
(u_p, v_p) \in \mathcal{C}([0,T);\, H^1(\mathbb{R}^2) \times H^3(\mathbb{R}^2)) \cap \mathcal{C}^1((0,T);\, H^1(\mathbb{R}^2) \times H^3(\mathbb{R}^2)) \\
\cap\, \mathcal{C}((0,T);\, H^2(\mathbb{R}^2) \times H^4(\mathbb{R}^2)).
\end{aligned}$$

Then, there exists some increasing continuous function $p(\cdot)$ independent of u_p and v_p, such that

$$\|(u_p(t), v_p(t))\|_{H^1 \times H^3} \le p(t + \|(u_{p,0}, v_{p,0})\|_{H^1 \times H^3}), \quad 0 \le t < T. \tag{2.7}$$

Proof. In the proof, we use another expression of the second equation of (P) with (US):

$$\begin{aligned}
&\frac{\partial v_p}{\partial t} = g\Delta v_p - v_p + P(u_p, v_p), \\
&P(u_p, v_p) = v_p + hv(u + v - 1)(1 - v) \\
&\qquad = h\left[v(1-v)u_p + \{1 + v(1-v)\}v_p\right].
\end{aligned} \tag{2.8}$$

Step 1. Multiply (2.8) by v_p and integrate the product in $\mathbb{R}^2$. From (B), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} v_p^2 dx + g\int_{\mathbb{R}^2} |\nabla v_p|^2\, dx \le C\left(\int_{\mathbb{R}^2} u_p^2\, dx + \int_{\mathbb{R}^2} v_p^2\, dx\right). \tag{2.9}$$

Multiply next (2.8) by Δv_p and integrate the product in $\mathbb{R}^2$. Then,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla v_p|^2\, dx + g\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + \int_{\mathbb{R}^2} |\nabla v_p|^2\, dx \le \frac{g}{2}\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + C\left(\int_{\mathbb{R}^2} u_p^2\, dx + \int_{\mathbb{R}^2} v_p^2\, dx\right).$$

It follows that

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla v_p|^2\, dx + g\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + 2\int_{\mathbb{R}^2} |\nabla v_p|^2\, dx \le C\left(\int_{\mathbb{R}^2} u_p^2\, dx + \int_{\mathbb{R}^2} v_p^2\, dx\right). \tag{2.10}$$

Multiply again (2.8) by $\Delta^2 v_p$ and integrate the product in $\mathbb{R}^2$. Then,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + g\int_{\mathbb{R}^2} |\nabla\Delta v_p|^2\, dx + \int_{\mathbb{R}^2} |\Delta v_p|^2\, dx \le \frac{g}{2}\int_{\mathbb{R}^2} |\nabla\Delta v_p|^2\, dx + \frac{1}{2g}\int_{\mathbb{R}^2} |\nabla P|^2 dx.$$

By (B) it is easy to see that $|\nabla P|^2 \le C\left(|\nabla u_p|^2 + |\nabla v_p|^2\right)$. Therefore, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + g\int_{\mathbb{R}^2} |\nabla\Delta v_p|^2\, dx + 2\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx \le C\left(\int_{\mathbb{R}^2} |\nabla u_p|^2\, dx + \int_{\mathbb{R}^2} |\nabla v_p|^2\, dx\right). \tag{2.11}$$

Meanwhile, multiply the first equation of (P) by u_p and integrate the product in $\mathbb{R}^2$. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} u_p^2 dx &+ a\int_{\mathbb{R}^2} |\nabla u_p|^2\, dx + (ce^{k\chi(1)} + d + f)\int_{\mathbb{R}^2} u_p^2 dx \\ &\le -b\int_{\mathbb{R}^2} u(1-u)\chi'(v)\nabla u_p \nabla v_p dx + c\tilde{v}\int_{\mathbb{R}^2} (e^{k\chi(\tilde{v})} - e^{k\chi(v)})u_p dx \\ &\le \frac{a}{2}\int_{\mathbb{R}^2} |\nabla u_p|^2\, dx + C\left(\int_{\mathbb{R}^2} u_p^2 dx + \int_{\mathbb{R}^2} |\nabla v_p|^2\, dx + \int_{\mathbb{R}^2} v_p^2 dx\right).\end{aligned}$$

So, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_p^2 dx + a\int_{\mathbb{R}^2} |\nabla u_p|^2\, dx \le C\left(\int_{\mathbb{R}^2} u_p^2 dx + \int_{\mathbb{R}^2} |\nabla v_p|^2\, dx + \int_{\mathbb{R}^2} v_p^2 dx\right). \tag{2.12}$$

By adding (2.9), (2.10) and (2.11) to (2.12) multiplied a large constant we obtain that with some constant $\delta > 0$

$$\|u_p(t)\|_{L^2}^2 + \|v_p(t)\|_{H^2}^2 \le Ce^{-\delta t}\|\Delta v_{p,0}\|_{L^2}^2 + p(t + \|u_{p,0}\|_{L^2} + \|v_{p,0}\|_{H^1}), \quad 0 \le t < T. \tag{2.13}$$

Step 2. Operate $\nabla\Delta$ to (2.8), take the inner product with $\nabla\Delta v_p$ and integrate the product in $\mathbb{R}^2$. Then,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\Delta v_p|^2\, dx + g\int_{\mathbb{R}^2} \left|\Delta^2 v_p\right|^2 dx + \int_{\mathbb{R}^2} |\nabla\Delta v_p|^2\, dx \le \frac{g}{2}\int_{\mathbb{R}^2} \left|\Delta^2 v_p\right|^2 dx + \frac{1}{2g}\int_{\mathbb{R}^2} |\Delta P|^2\, dx.$$

Thanks to (B) it is easily verified that

$$|\Delta P|^2 \le C\left(|\Delta u_p|^2 + |\Delta v_p|^2 + |\nabla v_p|^4 + |\nabla u_p|^2 |\nabla v_p|^2\right).$$

Moreover, we obtain

$$\int_{\mathbb{R}^2}\left(|\Delta v_p|^2+|\nabla v_p|^4\right)dx \le \|v_p\|_{H^2}^2+\|\nabla v_p\|_{L^4}^4 \le p(\|v_p\|_{H^2}) \tag{2.14}$$

and by noting that $\int_{\mathbb{R}^2}|\nabla u_p|^2\,dx \le \frac{1}{2}\int_{\mathbb{R}^2}\left(|\Delta u_p|^2+|u_p|^2\right)dx$,

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla u_p|^2|\nabla v_p|^2\,dx &\le \|\nabla u_p\|_{L^3}^2\|\nabla v_p\|_{L^6}^2\\ &\le p(\|u_p\|_{L^2}+\|v_p\|_{H^2})\|u_p\|_{H^2}^{\frac{4}{3}} \le C\,\|\Delta u_p\|_{L^2}^2+p(\|u_p\|_{L^2}+\|v_p\|_{H^2}).\end{aligned} \tag{2.15}$$

Therefore, it follows that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\Delta v_p|^2\,dx+g\int_{\mathbb{R}^2}\left|\Delta^2 v_p\right|^2dx+2\int_{\mathbb{R}^2}|\nabla\Delta v_p|^2\,dx&\\ \le C\,\|\Delta u_p\|_{L^2}^2+p(\|u_p\|_{L^2}+\|v_p\|_{H^2}).&\end{aligned} \tag{2.16}$$

Meanwhile, multiply the first equation of (P) by Δu_p and integrate the product in $\mathbb{R}^2$. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla u_p|^2\,dx&+a\int_{\mathbb{R}^2}|\Delta u_p|^2\,dx+(ce^{k\chi(1)}+d+f)\int_{\mathbb{R}^2}|\nabla u_p|^2\,dx\\ &\le -ck\int_{\mathbb{R}^2}e^{k\chi(v)}\chi'(v)u_p\nabla u_p\nabla v_p dx-b\int_{\mathbb{R}^2}\Delta u_p\nabla\left\{u(1-u)\nabla\chi(v)\right\}dx\\ &\qquad -c\tilde{u}\int_{\mathbb{R}^2}\{e^{k\chi(\tilde{v})}-e^{k\chi(v)}\}\Delta u_p dx\\ &\le \frac{a}{4}\int_{\mathbb{R}^2}|\Delta u_p|^2\,dx+\frac{b^2}{a}\int_{\mathbb{R}^2}\left|\nabla\left\{u(1-u)\nabla\chi(v)\right\}\right|^2dx\\ &\qquad +\frac{(ce^{k\chi(1)}+d+f)}{2}\int_{\mathbb{R}^2}|\nabla u_p|^2\,dx+p(\|v_p\|_{H^1}).\end{aligned}$$

In addition, by similar estimates to (2.14) and (2.15) we have

$$\int_{\mathbb{R}^2}\left|\nabla\left\{u(1-u)\nabla\chi(v)\right\}\right|^2dx \le \frac{a^2}{4b^2}\int_{\mathbb{R}^2}|\Delta u_p|^2\,dx+p(\|u_p\|_{L^2}+\|v_p\|_{H^2}).$$

It then follows that

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla u_p|^2\,dx+a\int_{\mathbb{R}^2}|\Delta u_p|^2\,dx+(ce^{k\chi(1)}+d+f)\int_{\mathbb{R}^2}|\nabla u_p|^2\,dx \le p(\|u_p\|_{L^2}+\|v_p\|_{H^2}).$$

We add this to (2.16) multiplied a small positive constant. Then, it is indicated from (2.13) that with some constant $\delta>0$

$$\|u_p(t)\|_{H^1}^2+\|v_p(t)\|_{H^3}^2 \le Ce^{-\delta t}\{\|\nabla u_{p,0}\|_{L^2}^2+\|\nabla\Delta v_{p,0}\|_{L^2}^2\}+p(t+\|u_{p,0}\|_{L^2}+\|v_{p,0}\|_{H^2}),\quad 0\le t<T.$$

Hence, we have proved the estimate (2.7). □

By Proposition 2.5, we can give the proof of global existence, Theorem 2.1. But, since quite similar arguments as [21, Proof of Theorem 1.1.] assure our global existence, we omit the proof.

3. Chemotactic Collapse of Mimura-Tsujikawa Model.

Let us assume that the domain $\Omega \subset \mathbb{R}^2$ is a bounded with smooth boundary and χ is a given positive constant. By using the same method of the previous sections we can show the existence of global solution to

$$(3.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla\{u(1-u)\nabla v\} + 1 - u \quad \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \alpha \Delta v + \beta u - \gamma v \quad \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x) \quad \text{in} \quad \Omega, \end{cases}$$

even if χ is sufficiently large (indeed, (B) is derived from (B_0), cf. [21, Theorem 3.4]). Meanwhile, Hillen and Painter treated the system consisted of:

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla\{u(1-u)\nabla v\} \quad \text{in} \quad \Omega \times (0,\infty),$$

and the same second equation, boundary and initial conditions as of (3.1), and showed the boundedness of solution, and then obtained the global existence for any $\chi > 0$ [13]. This system was introduced as a chemotaxis model with prevention of overcrowding cross diffusion. Then, $u(x,t)$ and $v(x,t)$ denote the population density of biological individuals and the concentration of chemical substance at a position $x \in \Omega$ and a time $t \in [0,\infty)$, respectively. Chemotaxis is the directed movement in a sense that the biological individuals have a tendency to move toward higher concentration of the chemical substance. As the chemotaxis effect the system has the advection term of the form negative diffusion. We then observe that the prevention of overcrowding cross-diffusion has some role of avoiding collapse even if the system has a linear decay or no growth term.

Let us consider the following reaction-diffusion-advection system having a simpler advection term:

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla(u \nabla v) + f(u) \quad \text{in} \quad \Omega \times (0,\infty),$$

and the same second equation, boundary and initial conditions as of (3.1). This system is Mimura-Tsujikawa model [2].

If the growth term $f(u)$ is a quadratic decay function such as $f(u) = u(1-u)$, then the global existence is assured and also the existence of exponential attractor [12] (as for other results on exponential attractor and pattern formation for Mimura-Tsujikawa model, refer [2, 28, 29, 30, 31]). We here note that it is well-known that if no growth term, that is, $f(u) \equiv 0$, the system is equivalent to Keller-Segel model, in which chemotactic collapse occurs for sufficiently large $\chi > 0$ (on the local and global existence and collapse for Keller-Segel model, refer [3, 4, 5, 6, 7, 8, 9, 10]).

Here, let us consider the problem for the following system consisted of

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla(u \nabla v) + 1 - u \quad \text{in} \quad \Omega \times (0,\infty),$$

which has a simple advection term and a linear decay growth, and the same second equation, boundary and initial conditions as of (3.1): does collapse occur or not for sufficiently large chemotactic coefficient χ? This is an open problem which we here does not treat directly. The similar arguments as for Keller-Segel model (e.g. [5]) permit to reduce the second equation to $0 = \Delta v + u - 1$ approximately for a case where $\frac{1}{|\Omega|}\int_\Omega u_0 dx = 1$, $|\Omega|$ being the measure of Ω, α is large and $\beta = \gamma = 1$. Then, to give an observation on collapse of Mimura-Tsujikawa model with a linear decay growth we shall consider the following system:

$$(3.2)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi\nabla(u\nabla v) + 1 - u & \text{in } \Omega\times(0,\infty), \\ 0 = \Delta v + u - 1 & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{in } \Omega. \end{cases}$$

Theorem 3.1. *Let the domain Ω be a ball with the radius 1, of which the center is the origin of $\mathbb{R}^2$. Then, for sufficiently large chemotactic coefficient $\chi > 0$ some radially symmetric solution $u(x,t)$ to (3.2) blows up in a finite time, that is, collapse can occur in the system (3.2).*

Proof. By simulating the result of Jäger and Luckhaus [5], we can construct the subsolution of u which blows up at the center of ball.

Introduce a function

$$U(t,\rho) = \int_{B_{\sqrt{\rho}}} (u-1)\,dx = \int_0^{\sqrt{\rho}} (u-1)\,rdr, \quad r = |x|, \quad 0 \le \rho \le 1,$$
$$B_{\sqrt{\rho}} = \{x \in \mathbb{R}^2;\ |x| \le \sqrt{\rho}\}.$$

Then by integrating the first equation of (3.2) we obtain

$$\int_{B_{\sqrt{\rho}}} \left\{\frac{\partial u}{\partial t} - \Delta u + \chi\nabla(u\nabla v) - (1-u)\right\} dx$$
$$= \frac{\partial U}{\partial t} - 4\rho\frac{\partial^2 U}{\partial \rho^2} - \chi\frac{\partial U^2}{\partial \rho} - (\chi-1)U = 0$$

with boundary conditions $U(t,0) = U(t,1) = 0$. Jäger and Luckhaus constructed a subsolution W, with parameters $0 < \rho_1 < \rho_2 < 1$, a, b and ρ_0 such that

$$W(t,\rho) = \begin{cases} \dfrac{a\rho}{\rho+\tau^3}, & 0 < \rho < \rho_1, \\ \gamma\left(1 - \rho - \dfrac{(\rho_2-\rho)_+^2}{\rho_2}\right), & \rho_1 \le \rho < 1, \end{cases}$$

where $\tau = \rho_0 - bt$ and $\gamma = \left(1 - \rho_1 - (\rho_2-\rho_1)^2/\rho_2\right)^{-1} \frac{a\rho_1}{\rho_1+\tau^3}$, with boundary conditions $W(t,0) = W(t,1) = 0$. Indeed, we obtain

$$\frac{\partial W}{\partial t} - 4\rho\frac{\partial^2 W}{\partial \rho^2} - \chi\frac{\partial W^2}{\partial \rho} - (\chi-1)W$$
$$= \begin{cases} \left\{\dfrac{3b\tau^2}{\rho+\tau^3} + 2(4-a\chi)\dfrac{\tau^3}{(\rho+\tau^3)^2} - (\chi-1)\right\} W, & 0 < \rho < \rho_1, \\ \left\{\dfrac{2b\rho_0^2}{\rho_1} + \dfrac{8}{\rho_2(1-\rho_2)} + 1 - \chi\left(1 - \dfrac{2a}{1-\rho_2}\right)\right\} W, & \rho_1 \le \rho < 1. \end{cases}$$

Choosing the parameters satisfying that $b\rho_0^2$ sufficiently small, a small as $4 - a\chi > 0$ and $1 - \frac{2a}{1-\rho_2} > 0$, and χ sufficiently large, then the coefficients of W of the right hand side are negative, hence, provided with the initial functions as $W(0,\rho) \le U(0,\rho)$, the comparison is possible. This implies that collapse occurs at the center of Ω. □

ACKNOWLEDGMENTS

The author would like to express his hearty thanks to Ministry of Education, Cultrure, Sports, Science and Technology of Japan for financial support Grant-in-Aid for Young Scientists (B) No.16740104.

References

[1] M. Hildebrand, M. Kuperman, H. Wio, A. S. Mikhailov, and G. Ertl, *Self-Organized Chemical Nanoscale Microreactors*, Physical Review Letters **83** (1999), 1475–1478.

[2] M. Mimura and T. Tsujikawa, *Aggregating Pattern Dynamics in a Chemotaxis Model Including Growth*, Physica A **230** (1996), 499–543.

[3] H. Gajewski and K. Zacharias, *Global Behavior of a Reaction-Diffusion System Modeling Chemotaxis*, Math. Nachr. **195** (1998), 77–114.

[4] M. A. Herrero and J. J. L. Velázquez, *A Blow-up Mechanism for a Chemotaxis Model*, Ann. Scoula Norm. Sup. Pisa IV **35** (1997), 633–683.

[5] W. Jäger and S. Luckhaus, *On Explosions of Solutions to a System of Partial Differential Equations Modelling Chemotaxis*, Trans. AMS **329** (1992), 819–824.

[6] T. Nagai, *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domain*, J. of Inequal. and Appl. **6** (2001), 37–55.

[7] T. Nagai, T. Senba and T. Suzuki, *Chemotactic Collapse in a Parabolic System of Mathematical Biology*, Hiroshima Math. J. **30** (2000), 463–497.

[8] T. Nagai, T. Senba and K. Yoshida, *Application of the Trudinger-Moser Inequality to a Parabolic System of Chemotaxis*, Funkcial. Ekvac. **40** (1997), 411–433.

[9] K. Osaki and A. Yagi, *Finite Dimensional Attractor for One-Dimensinal Keller-Segel Equations*, Funkcial. Ekvac. **44** (2001), 441–469.

[10] T. Senba and T. Suzuki, *Parabolic System of Chemotaxis: Blow Up in a Finite and the Infinite Time*, Methods Appl. Anal. **8** (2001), 349–368.

[11] A. Yagi, *Norm Behavior of Solutions to the Parabolic System of Chemotaxis*, Math. Japonica **45** (1997), 241–265.

[12] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, *Exponential Attractor for a Chemotaxis-Growth System of Equations*, Nonlinear Analysis: Theory, Methods and Applications **51** (2002), 119–144.

[13] T. Hillen and K. Painter, *Global Existence for a Parabolic Chemotaxis Model with Prevention of Overcrowding*, Adv. Appl. Math. **26** (2001), 280–301.

[14] M. Hildebrand, M. Ipsen, A. S. Mikhailov and G. Ertl, *Localized Nonequilibrium Nanostructures in Surface Chemical Reactions*, New J. Phys., **5** (2003), 61.1–61.28.

[15] M. Eiswirth and G. Ertl, *Pattern Formation on Catalytic Surfaces, in "Chemical Waves and Patterns", eds. R. Kapral and K. Showalter*, 447–483, Kluwer, Dordrecht, 1995.

[16] T. Tsujikawa and A. Yagi, *Exponential Attractor for an Adsorbate-Induced Phase Transition Model*, Kyushu J. Math. **56** (2002), 313–336.

[17] Y. Takei, K. Osaki, T. Tsujikawa and A. Yagi, *Exponential Attractor for an Adsorbate-Induced Phase Transition Model with Periodic Boundary Conditions*, Differential Equations and Applications vol. 4, ed. Yeol Je Cho, Nova Science Publishers, to appear.

[18] A. Eden, C. Foias, B. Nicolaenko, and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Research in Applied Mathematics vol. 37, John-Wiley and Sons, New York, 1994.

[19] Y. Takei, T. Tsujikawa and A. Yagi, *Numerical Computations and Pattern Formation for Adsorbate-Induced Phase Transition Model*, Sci. Math. Japonicae, to appear.

[20] T. Tsujikawa and A. S. Mikhailov, *Singular Limit Analysis of an Adsorbate-Induced Phase Transition Model*, preprint.

[21] K. Osaki, Y. Takei and T. Tsujikawa, *Global Solution to a Reaction Diffusion Phase Transition System in* $\mathbb{R}^2$, Adv. Math. Sci. Appl. **14** (2004), 559–576.

[22] J. W. Cholewa and T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, London Mathematical Society Lecture Note Series, vol. 278, Cambridge University Press, 2000.

[23] H. Hoshino and Y. Yamada, *Solvability and Smoothing Effect for Semilinear Parabolic Equations*, Funkcial. Ekvac. **34** (1991), 475–494.

[24] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and Their Applications vol. 16, Birkhäuser, Basel, 1995.

[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences 44, Springer-Verlag, New York, 1983.

[26] H. Tanabe, *Equation of Evolution*, Iwanami, Tokyo, 1975 (in Japanese); English translation, Pitman, London, 1979.

[27] K. Osaki and A. Yagi, *Global Existence for a Chemotaxis-Growth System in* $\mathbb{R}^2$, Adv. Math. Sci. Appl. **12** (2002), 587–606.

[28] M. Aida, M. Efendiev and A. Yagi, *Quasilinear Abstract Parabolic Evolution Equations and Exponential Attractors*, Osaka J. Math., to appear.

[29] M. Aida, K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, *Chemotaxis and Growth System with Singular Sensitivity Function*, Nonlinear Analysis: Real World Applications **6** (2005), 323–336.

[30] M. Aida and A. Yagi, *Target Pattern Solutions for Chemotaxis-Growth System*, Math. Sci. Japonicae, to appear.

[31] M. Mimura, T. Tsujikawa, R. Kobayashi and D. Ueyama, *Dynamics of Aggregating Patterns in a Chemotaxis-Diffusion-Growth Model Equation*, Forma **8** (1993), 179–195.