Bifurcation Problem for Eigenvalues of a Nonlocal Differential Equation

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Abstract. Bifurcation problem for the eigenvalues of a second order differential equation with nonlocal term is considered. The problem is from a linearized stability problem for the kinematic equation with global feedback [1]. By differentiating the equation a third order equation is derived, and then the bifurcation curves of eigenvalues are investigated.

Key words: nonlocal term, bifurcation problem, eigenvalue, stability, kinematic equation.

1. Eigenvalue Problem with Nonlocal Term.

In this report we consider a second order eigenvalue problem with nonlocal term:

$$(1.1)\qquad \begin{cases} X'' + h\displaystyle\int_0^x X(\xi)\,d\xi = \mu X, & 0 < x < \pi; \\ X(0) = X(\pi) = 0, \end{cases}$$

where $h \in \mathbb{R}$ is a parameter. The problem is to consider the existence and sign of μ. Nonlocal partial differential equations appear as nonlocal coupling systems in physics, chemistry and so on. The equation (1.1) is derived from a linearized stability problem [1] for the kinematic equation with global feedback coupling [2, 3], which is a model equation for BZ reaction [4]. The simplified equation is denoted as:

$$(1.2)\qquad \begin{cases} u_t = Du_{xx} + h\displaystyle\int_0^x u(\xi)\,d\xi, & t > 0,\ 0 < x < \pi; \\ u(t,0) = u(t,\pi) = 0, & t > 0; \\ u(0,x) = u_0(x), & 0 < x < \pi. \end{cases}$$

where $D > 0$ is a diffusion coefficient. To investigate the stability of zero solution to (1.2) put the solution as

$$u(t,x) = e^{\mu t}X(x),$$

and denote μ/D and h/D as again μ and h, then the equation (1.1) is obtained.

2. Bifurcation Curves of Eigenvalues and Stability of Solution.

Consider the equation (1.1). Let us put

$$Y(x) = \int_0^x X(\xi)\,d\xi, \qquad 0 < x < \pi.$$

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Then, we have a third order differential equation:

$$(2.1)\qquad \begin{cases} Y''' - \mu Y' + hY = 0, & 0 < x < \pi; \\ Y(0) = 0, \quad Y'(0) = Y'(\pi) = 0. \end{cases}$$

Consider the corresponding algebraic equation:

$$(2.2)\qquad \lambda^3 - \mu\lambda + h = 0.$$

By Cardano's formula the solutions to (2.2) are represented as:

$$\lambda = 2\alpha, \quad -\alpha \pm \sqrt{3}\beta i,$$

where

$$(2.3)\qquad 2\alpha = \sqrt[3]{\frac{-9h + \sqrt{81h^2 - 12\mu^3}}{18}} - \sqrt[3]{\frac{9h + \sqrt{81h^2 - 12\mu^3}}{18}},$$

$$(2.4)\qquad 2\beta = \sqrt[3]{\frac{-9h + \sqrt{81h^2 - 12\mu^3}}{18}} + \sqrt[3]{\frac{9h + \sqrt{81h^2 - 12\mu^3}}{18}}.$$

Hence, the solution to (2.1) is denoted by

$$Y(x) = C_1 e^{2\alpha x} + e^{-\alpha x}\left(C_2 \cos\sqrt{3}\beta x + C_3 \sin\sqrt{3}\beta x\right)$$

with constants C_i, $i = 1, 2, 3$. Since it is easy to see that

$$Y'(x) = 2C_1\alpha e^{2\alpha x} - C_2 e^{-\alpha x}\left(\alpha\cos\sqrt{3}\beta x + \sqrt{3}\beta\sin\sqrt{3}\beta x\right) - C_3 e^{-\alpha x}\left(\alpha\sin\sqrt{3}\beta x - \sqrt{3}\beta\cos\sqrt{3}\beta x\right),$$

a system of equations is derived from the boundary conditions:

$$\begin{cases} C_1 + C_2 = 0, \\ 2\alpha C_1 - \alpha C_2 + \sqrt{3}\beta C_3 = 0, \\ 2\alpha e^{3\alpha\pi} C_1 - \left(\alpha\cos\sqrt{3}\beta\pi + \sqrt{3}\beta\sin\sqrt{3}\beta\pi\right) C_2 - \left(\alpha\sin\sqrt{3}\beta\pi - \sqrt{3}\beta\cos\sqrt{3}\beta\pi\right) C_3 = 0. \end{cases}$$

Taking $C_i \neq 0$, $i = 1, 2, 3$, into account, we obtain an equation for the eigenvalues:

$$(2.5)\qquad F(\mu, h) := 2\alpha\beta\left(e^{3\alpha\pi} - \cos\sqrt{3}\beta\pi\right) + \sqrt{3}(\alpha^2 + \beta^2)\sin\sqrt{3}\beta\pi = 0$$

with (2.3) and (2.4). Solving this equation by numerical simulation, it shows that there exists a threshold $h_{\mathrm{th}}(\fallingdotseq 0.8854)$ such that for $h < h_{\mathrm{th}}$ the eigenvalues μ_n, $n = 1, 2, 3, \cdots$, are all negative, while for $h \geq h_{\mathrm{th}}$ first eigenvalue μ_1 is nonnegative (see Figures 1 and 2). Therefore, it concludes that

$$\text{the zero solution to (1.2) is } \begin{cases} \text{stable for } h \leq Dh_{\mathrm{th}}, \\ \text{unstable for } h > Dh_{\mathrm{th}}. \end{cases}$$

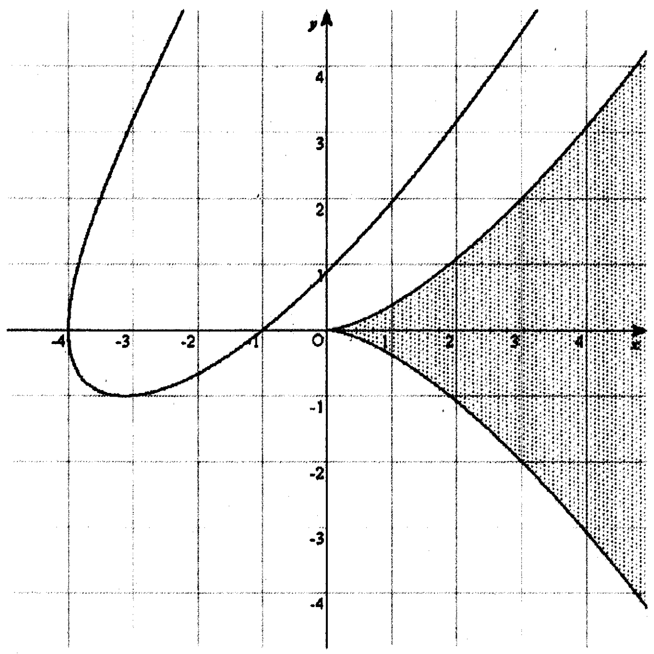

Figure 1: Implicit plot of the equation $F(x, y) = 0$, $-5 \le x \le 5$, (blue line). The curve does not exist in the area: $81y^2 - 12x^3 < 0$ (red area).

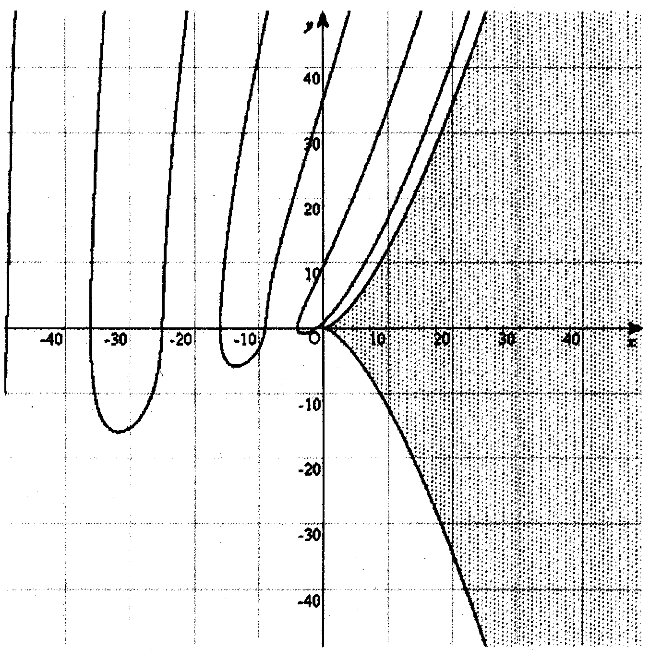

Figure 2: Implicit plot of the equation $F(x, y) = 0$, $-50 \le x \le 50$, (blue lines). The curves do not exist in the area: $81y^2 - 12x^3 < 0$ (red area).

3. Appendix.

By the above result we have known the existence of threshold $h_{\rm th}$ for the stability of zero solution to (1.2). We shall compare the result with the usual Fourier mode analysis. By the boundary condition of (1.1) let us put the solution X to (1.1) as

$$X(x) \sim \sum_{n=1}^{\infty} a_n \sin nx, \quad 0 < x < \pi. \tag{3.1}$$

Then, multiplying $\sin nx$ to the equation of each n and integrating it over $(0,\pi)$, we have

$$-n^2 \int_0^{\pi} \sin^2 nx\, dx + \frac{1}{n}\int_0^{\pi} (1-\cos nx)\sin nx\, dx = \tilde{\mu}\int_0^{\pi} \sin^2 nx\, dx, \quad n = 1, 2, 3, \cdots.$$

Therefore, it derives that

$$\tilde{\mu}_n = -n^2 + \frac{2h\{1-(-1)^n\}}{n^2\pi}, \quad n = 1, 2, 3, \cdots.$$

This also shows that there exists a threshold $\tilde{h}_{\rm th} = \frac{\pi}{4} \fallingdotseq 0.7854$ such that for $h < \tilde{h}_{\rm th}$ all $\tilde{\mu}_n$, $n = 1, 2, 3, \cdots$, are negative and for $h \geq \tilde{h}_{\rm th}$ nonnegative.

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