On a remark of quasi-continuous modules and χ-extending modules

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Abstract

In section 1, for an extending module M, we shall give a necessary and sufficient condition of M to be a quasi-continuous modules.

In section 2, we shall give a sufficient condition for which Type 1-χ-extending module and Type 2-χ-extending module are the same.

Key words : ring, module, extending, quasi-continuous

Introduction

Throughout this note all rings are associative with identity and all modules are unital right modules.

Let R be a ring and let M be an R-module. We use the symbol $N \subseteq_e M$ to mean that N is essential submodule of M. For any submodule N of M, a closure of N (in M) is a submodule K of M which is maximal in the collection of submodules of M containing N as an essential submodule. A submodule K of M is called closed (in M) if K has no proper essential extension in M. Given any submodule N of M, a complement of N (in M) means a submodule L of M which is maximal in the collection of submodules H with the property $H \cap N = 0$.

A submodule L is called a complement (in M) if there exists a submodule N of M such that L is a complement of N. It is well known that a submodule K of M is closed if and only if K is a complement in M.

A module M is called an extending module if every closed submodule of M is a direct summand of M and M is called quasi-continuous if M is an extending module and for any direct summands A, B of M with $A \cap B = 0$, $A \oplus B$ is direct summand of M. For extending modules and quasi-continuous modules, the reader is referred to [2],[3].

In [4] , Dogrouoz and Smith mention χ-extending modules for a class χ of R-modules , which contains 0 and is closed under isomorphic image.

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However , in the present paper, we consider certain χ-extending modules for any class χ of R-modules. A submodule N of an R-module is called a χ-submodule if $N \in \chi$. We shall say that an R-module M is Type 1-χ-extending if for any x-submodule N of M, every complement of N in M is a direct summand of M. On the other hand , an R-module M is Type 2-χ-extending if for any χ-submodule N of M, every closure of N in M is a direct summand.

Remark 1 In general , Type 1-χ-extending module is not Type 2 . (See:[4] Examples 2.7, 2.11)

Remark 2 For χ = the class of all R-modules, following statements are equivalent for an R-module M.

(1) M is extending.

(2) M is Type 1-χ-extending.

(3) M is Type 2 - χ-extending.

Section 1. An equivalent condition

In this section , we shall mention a condition for which the extending module becomes quasi-continuous.

Proposition 1 Let M be an extending module. Following statements are equivalent.

(1) If M' is a direct summand of M, and if A and B are direct summands of M' with $A \cap B = 0$ and $A \oplus B \leq_e M'$, then $A \oplus B = M'$.

(2) M is quasi-continuous.

Proof. (1)$\Rightarrow$(2). Let A and B be any direct summands of M with $A \cap B = 0$; $A \oplus X = M$ and $B \oplus Y = M$, for some submodules X, Y of M.
And, since M is extending, there is a direct summand M' of M such that $A \oplus B \leq_e M'$. From the modular law, $A \oplus (X \cap M') = M'$ and $B \oplus (Y \cap M') = M'$. That is, A and B are direct summands of M'. By (1), $A \oplus B = M'$ ($\leq \oplus M$). Hence M is quasi-continuous.

(2)$\Rightarrow$(1). Let M be quasi-continuous and M' a direct summand of M. And, let A and B are any direct summands of M' with $A \cap B = 0$ and $A \oplus B \leq_e M'$, As well known, M' is quasi-continuous.
Hence , $A \oplus B \leq \oplus M'$. As $A \oplus B \leq_e M'$, it follows $A \oplus B = M'$.

Section 2 Type 1 and Type 2 - χ -extending modules

In Remark 1, we noted that Type 1- χ -extending module is not same with Type 2 , in general. In this section we shall study the problem when Type 1 is Type 2.

Let χ be a class of submodules of a module M . We consider the following condition (*); For submodules N and X of M with $N \in X$, $N \oplus X \leq_e M$, implies $X \in \chi$.

Proposition 2 Assume χ satisfies (*), then the following statements are equivarent.

(1) M is Type 1- χ -extending.

(2) M is Type 2- χ -extending.

Proof. (1)$\Rightarrow$(2). Let $N \in \chi$, and let K be a closure of N . Then K is a closed submodule of M , that is , complement submodule of M . There exist a submodule X of M such that K is a complement of X . Therefore, $X \oplus K \leq_e M$ and $X \oplus N \leq_e M$. From the condition (*) , $X \in \chi$. So we see that K is a direct summand of M .

(2)$\Rightarrow$(1). Let N be a χ -submodule of M and let L be a complement of N . Since L is a closed submodule, the closure of L is L and $N \oplus L \leq_e M$. Then $L \in \chi$. Hence, by hypothesis, L is a direct summand of M .

References

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