On The Noetherian Ideal Theory

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Introduction. The additive ideal theory of the noncommutative ring with the ascending chain condition (a.c.c.) for two-sided ideals has been studied by many authors. Recently, K. L. Chew [4] has obtained new results.

In this paper we have two purposes. One of them is to examine principal theorems, obtained by Chew and by other authors, on the following conditions which are weaker than the a.c.c. :

(F). *Every ideal of a ring R is represented as an intersection of a finite number of* $\cap$*-irreducible ideals.*

(K). *For any elements a and b, (a)(b) is a finitely generated ideal.* (K. Murata [6]).

The other is to obtain new conditions necessary and sufficent for a ring R with (F) and (K) to have the property of making any ideal of R be represented as an intersection of a finite number of primary ideals. In such a case it is said in this paper that a ring *R has the Noetherian ideal theory*.

In §1, on the condition (F) we shall examine ([4] 3.3. Theorem). In §2, on the conditions (F) and (K) we shall examine theorems in [2], [3] and [4]. In §3, we shall seek for the conditions which are equivalent to those that a ring R with (F) and (K) has the Noetherian ideal theory.

§1. We use R as a noncommutative ring which has not necessarily a unit element. Unless especially mentioned, A, B,…… mean ideals of R and a, b,……, x, y, z mean elements. The term "ideal" always means "two-sided ideal" and $(x, y, \cdots\cdots)$ means the ideal generated by elements x, y,……. We set $A:B=\{x \mid (x)B\subseteq A\}$, but we put down instead of $A:(x)$ $A:x$ for a simplicity.

Definition 1.1. An element x is said *not to be (right) prime to A (nrp to A)* if $A:x\neq A$. Otherwise x is said *to be (right) prime to A (rp to A)*. An ideal B is said *to be nrp to A* if every element x in B is nrp to A.

Definition 1.2. The set $\{x \mid (x)^n\subseteq A$ for some positive integer $n\}$, denoted by $\bar{A}$, is called *the nilpotent radical of A*. It is easily proved that $\bar{A}$ is an ideal. Further an ideal $\cap\{P \mid P$ is a prime divisor of $A\}$, denoted by $\tilde{A}$, is called *the McCoy's radical of A*.

Definition 1.3. An ideal A is said *to be (right) primary with respect to the nilpotent (resp. McCoy's) radical* if $A:x\neq A$ implies $x\in\bar{A}$ (resp. $x\in\tilde{A}$).

We shall weaken the concept of the so-called Noetherian ideal theory as follows :

Definition 1.4. R is said *to have the Noetherian ideal theory* if every ideal of R is represented as an intersection of a finite number of primary ideals.

Definition 1.5. The set $\{x \mid (A:x)\cap Y=A$ implies $Y=A\}$, denoted by ter(A), is called *the tertiary radical of A*. A is said *to be tertiary* if $A:x\neq A$ implies $x\in ter(A)$. It is well known that for any ideal A, ter(A) is a ideal.

Clearly a $\cap$-irreducible ideal is tertiary. Next, we shall treat the nilpotent radical as the radical of ideals and examine ([4], 3.3. Theorem).

Theorem 1.1. *Let the radical of ideals be the nilpotent radical. In R with* (F) *the following statements are equivalent to one another.*

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(1). *R has the Noetherian ideal theory.*

(2). *Any* $\cap$*-irreducible ideal is primary.*

(3). *For any ideal A and for any element b, there exists a positive integer n such that* $A \cap (b)^n \subseteq A(b)$.

(4). *For any ideal A,* $ter(A) \subseteq \tilde{A}$.

(5). *Any teriary ideal is a primary ideal.*

Furthermore, the following statement (6) *implies each of the statements* (1)—(5).

(6). *For any ideal A and for any element b, there exists a positive integer n such that* $A = (A + (b)^n) \cap (A : (b)^n)$.

Proof. (1)$\Longleftrightarrow$(2) : This is evident from (F).

(1)$\Longrightarrow$(3) : If we mind that $b \in \tilde{A}$ implies $(b)^n \subseteq A$ for some positive integer n, this is proved in a way similar to McCarthy [1].

(3)$\Longrightarrow$(4) : This is proved in a way similar to the proof of ([4], 3.1. Lemma).

(4)$\Longrightarrow$(5) : This is evident.

(5)$\Longrightarrow$(1) : From (F) we have $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ where each Q_i is $\cap$-irreducible, hence Q_i is tertiary and from (5) Q_n is primary.

(6)$\Longrightarrow$(3) : This is proved in a way similar to the proof of ([4], 3.2. Lemma).

Corollary. *In R with* (F) *and* (K), *we have the following:*

(i). *Statemente* (1)—(6) *in Theorem* 1.1 *are equivalent to ene another.*

(ii). *In Theorem* 1.1, *we can replace* (3) (*resp.* (6)) *with following* (3′) (*resp.* (6′)) :

(3′). *For any ideal A, for any finitely generated ideal N, and for any positive integer k, there exists a positive integer n such that* $A \cap N^{nk} \subseteq AN^k$.

(6′). *For any ideal A and any finitely generated ideal N, there exists a positive integer n such that* $A = (A + N^n) \cap (A : N^n)$.

(iii) *Furthermore, if R has the Noetherian ideal theory then for any finitely generated ideal N there exists a positive integer n such that* $N^n \cap (0 : N^n) = 0$.

When we treat the McCoy′s radical as the radical of ideals, the following theorem holds.

Theorem 1.2. *Let the radical of ideals be the McCoy′s one. In R with* (F) *the following statements are equivalent to one another.*

(1). *R has the Noetherian ideal theory.*

(2). *Any* $\cap$ *-irreducible ideal is primary.*

(3). *For any ideals A and B, there exists an ideal B′ such thet* $\tilde{B} = \tilde{B}'$ *and* $A \cap B' \subseteq AB$.

(4). *For any ideal A,* $ter(A) \subseteq \tilde{A}$.

(5). *Any tertiary ideal is primary.*

Furthermore, the following statement (6) *implies each of the statements* (1)—(5).

(6). *For any ideals A and B, there exists an ideal B′ such that* $\tilde{B} = \tilde{B}'$, $B \supseteq B'$ *and* $A = (A + B') \cap (A : B')$.

Proof. (1)$\Longleftrightarrow$(2) : This is evident.

(1)$\Longrightarrow$(3) : From (1), $AB = \bigcap_{k=1}^{n} Q_k$ where Q_k is primary ($k = 1, \cdots, n$), For each k, $AB \subseteq Q_k$, then $A \subseteq Q_k$ or $B \subseteq \tilde{Q}_k$. We may assume that $A \subseteq Q_i$ ($i = 1, \cdots, r$) and that $B \subseteq \tilde{Q}_j$ ($j = r+1, \cdots, n$). Now let $B' = B \cap (\bigcap_{j=r+1}^{n} Q_j)$, then $\tilde{B}' = \tilde{B} \cap (\bigcap_{j=r+1}^{n} \tilde{Q}_j) = \tilde{B}$, $B' \subseteq B$ and $B' \subseteq \bigcap_{j=r+1}^{n} Q_j$. Thus $A \cap B' \subseteq (\bigcap_{i=1}^{r} Q_i) \cap (\bigcap_{j=r+1}^{n} Q_j)$

$=AB$.

(3)$\Longrightarrow$(2) : Let A be $\cap$-irreducible and let A be not primary, then there exists an element b such that $A:b \subsetneqq A$ and $b \notin \tilde{A}$. From (3), there exists an ideal B' such that $\tilde{B}'=(\tilde{b})$ and $(A:b)\cap B' \subseteq (A:b)\cdot(b) \subseteq A$. Hence $A=(A+B')\cap(A:b)$. But $A:b \subsetneqq A$ and $A+B' \subsetneqq A$. For if $A+B' \subseteq A$, then $\widetilde{A+B'} \subseteq \tilde{A}$, hence $b \in (\tilde{b})=\tilde{B}' \subseteq \widetilde{A+B'} \subseteq \tilde{A}$. Thus $b \in \tilde{A}$. This is a contradiction.

(3)$\Longrightarrow$(4) : Let x be an element such that $(A:x)\cap B=A$ implies $B=A$ for any ideal B. From (3) there exists an ideal X' such that $\tilde{X}'=\widetilde{(x)}$ and $A=(A:x)\cap(A+X')$. Then $A+X'=A$ and hence $X' \subseteq A$. Thus $\widetilde{X'} \subseteq \tilde{A}$ and so we have $(x) \subseteq \tilde{A}$.

(4)$\Longrightarrow$(5) and (5)$\Longrightarrow$(1) : These are evident.

(6)$\Longrightarrow$(3) : For any ideals A and B, from (6) there exists an ideal B' such that $B' \subseteq B$, $\tilde{B}'=\tilde{B}$ and $AB=(AB+B')\cap(AB:B') \supseteq (AB+B')\cap(AB:B) \supseteq B'\cap A$.

§**2.** In this section we assume the condition (K) mentioned in the introduction. Of course if R has the a. c. c., then R has (F) and (K). Now we shall show that in R with (K) the nilpotent radical coincides with the McCoy's one.

Lemma 2.1. *For any ideal A, $\bar{A}=\bar{\bar{A}}$ if and only if $\bar{A}$ is a semi prime ideal.*

Proof. If $\bar{A}=\bar{\bar{A}}$ and $\bar{A}$ is not semi prime, there exists an element b such that $b \notin \bar{A}$ and $(b)^2 \subseteq \bar{A}$. But $b \in \bar{\bar{A}}=\bar{A}$. This is a contradiction. Conversely if $\bar{A}$ is semi prime and $b \in \bar{\bar{A}}$, then there exists a positive integer n such that $(b)^n \subseteq \bar{A}$. From the assumption of $\bar{A}$, $b \in \bar{A}$.

Theorem 2.1. *For any ideal A, $\bar{A}=\bar{\bar{A}}$ if and only if $\bar{A}=\tilde{A}$.*

Proof. "*if part*" is evident. As to "*only if part*" $\bar{A}$ is semi prime from Lemma 2.1 Then $\bar{A}$ is an intersection of prime ideals. On the other hand, $\bar{A} \subseteq \tilde{A}=\cap\{P \mid P$ covers all primes containing $A\}$. Therefore we obtain easily $\bar{A}=\tilde{A}$.

Lemma 2.2. *In R with (K), for any ideal A, $\bar{A}=\bar{\bar{A}}$.*

Proof. Easily $\bar{A} \subseteq \bar{\bar{A}}$. Conversely, if $x \in \bar{\bar{A}}$, then $(x)^n \subseteq \bar{A}$ for some positive integer n. From (K), $(x)^n$ is finitely generated, hence $((x)^n)^m \subseteq A$ for some positive integer m. Since $(x)^{nm} \subseteq A$, $x \in \bar{A}$.

Corollary. *In R with (K), for any ideal A $\bar{A}=\tilde{A}$.*

Definition 2.1. An ideal A is called a (*right*) *primal ideal* if the set $A^*=\{x \mid x$ is nrp to $A\}$ is an ideal.

Henceforth, we shall denote the radical of A with $\bar{A}$, and use A^* in this sense.

Definition 2.3. A set M of elements of R is called *an m-system* if for any elements x, y in M there exists an element r in R such that $xry \in M$. The null set is also, by the definition, an *m*-system.

Lemma 2.3. *In R with* (K), *for any tertiary ideal A the set $M=\{x \mid x$ is rp to $A\}$ is an m-system.*

Proof. We may assume that $M \neq \emptyset$. We shall denote the complement of M with M^c. Since a tertiary ideal is primal, M^c is an ideal. If M is not an *m*-system, there exist two elements x, y in M such that $xRy \subseteq M^c$. From ([3], §8, Lemma 1) there exists an element z in M such that $(z)(y) \subseteq M^c$. From (K) we have $(z)(y)=(t_1,\cdots,t_n)$, then for each i t_i is nrp to A, hence $A:t_i \supsetneqq A$. Therefore $A:((z)(y))=A:(t_1,\cdots,t_n)=\bigcap_{i=1}^{n}(A:t_i) \supsetneqq A$ by the tertiarity of A. On the other hand, $A:((z)(y))=(A:y):z=A:z=A$. This is a contradiction.

Definition 2.3. If an ideal A can be expressed in the form $A=T_1\cap\cdots\cap T_n$, where each T_i is a tertiary (resp. primary) ideal, we shall say that A has a *tertiary* (*resp. primary*) *decomposition* and the individual T_i will be called *tertiary* (*resp. primary*) *components* of the decomposition. A decomposition, in which no T_i contains the intersection of the remaining T_j, is said *to be* *irredundant*. An irredundant tertiary (resp. primary) decomposition, in which the tertiary (resp. nilpotent) radicals of various components are all different, is called *a normal decomposition*.

Theorem 2.2. *In R with (F) and (K), any ideal A has a normal tertiary decomposition*

(2.1) $A = T_1 \cap \cdots \cap T_n$,

where $ter(T_i)$ *is a prime ideal for each i. In any two normal tertiary decompositions of A, the number of tertiary components is the same and the radicals of the two sets of tertiary components coincide with each other. We shall call them the associated primes of A. Further,* $ter(A) = ter(T_1) \cap \cdots \cap ter(T_n)$.

Proof. From (F), we have

(2.2) $A = Q_1 \cap \cdots \cap Q_r$,

where Q_i is $\cap$-irreducible, hence for each i, Q_i is tertiary, and from Lemma 2.3 ter $(Q_i) = Q_i^*$ is prime. The rest is obtained from (Y. Kurata, [7], Lemma 1.7, Lemma 1.8 and Theorem 2.2).

Corollary. *In R with* (F) *and* (K), *for any ideal A,* $ter(A) \supseteq \bar{A}$. *Hence any primary ideal is tertiary ideal.*

It is easily proved that an ideal Q is primary if and only if $Q^* = \bar{Q}$. Hence we have the following.

Lemma 2.4. *In R with* (F) *and* (K), *the radical ideal of any primary ideal Q is prime.*

Proof. Q is tertiary, then from Lemma 2.3 $\bar{Q} = Q^*$ is prime.

Remark. Lemma 2.4 holds in R with only (K). And also the following Lemma holds in R with only (K). But we shall treat R with both (F) and (K) for simplicity.

Lemma 2.5. *In R with* (F) *and* (K), *let* Q_1 *and* Q_2 *be two primary ideals such that* $\bar{Q}_1 = \bar{Q}_2 = P$ *and let* $Q = Q_1 \cap Q_2$, *then Q is primary and* $\bar{Q} = P$.

Proof. Since Q_1 and Q_2 are tertiary, $Q_i^* = \bar{Q}_i = ter(Q_i) = P$ $(i=1,2)$. From ([7], Lemma 1.8) Q is tertiary, hence $Q^* = ter(Q) = P = \bar{Q}$. Thus Q is primary.

In R with (F) and (K), let

(2.3) $A = A_1 \cap \cdots \cap A_r$

be a primary decomposition, then from Lemma 2.5, (2.3) can be refined into the following normal primary decomposition

(2.4) $A = Q_1 \cap \cdots \cap Q_n$.

Since for each i $ter(Q_i) = \bar{Q}_i$, (2.4) is a normal tertiary decomposition. Therefore we obtain the following theorem:

Theorem 2.3. *In R with* (F) *and* (K), *from any primary decomposition of A a normal primary decomposition is refined; and it is a normal tertiary decomposition, too. In any two normal primary decompositions of A the number of primary components is the same and the radicals of the two sets of primary components coincide with one another.*

Definition 2.4. For any set M, we shall define the (*right*) *upper M-component* of A, denoted by $u(A, M)$, as follows: For $M = \phi$ let $u(A, M) = A$, and for $M \neq \phi$ let $u(A, M)$ *be the intersection of all ideals which contain A, and that are such that every element in M is rp to them.*

Definition 2.5. For any m system M, we shall define the (*right*) *lower M-component of A*, denoted by $l(A, M)$, as follows: For $M = \phi$ let $l(A, M) = A$. for $M \neq \phi$ let $l(A, M) = \{x \mid (x)(m) \subseteq A$ *for some* $m \in M\}$. It is easily proved that $l(A, M)$ is an ideal.

We know the following lemma from [3].

Lemma 2.6. *For any set M, every element in M is rp to* $u(A, M)$.

Lemma 2.7. $l(A, M) = \{x \mid xRm \subseteq A$ *for some* $m \in M\}$.

Proof. Let $L' = \{x \mid xRm \subseteq A$ for some $m \in M\}$ and let $l(A, M) = L$. If $(x)(m) \subseteq A$ for some m in M, then $xRm \subseteq A$, and hence $L \subseteq L'$. Conversely, let $yRm \subseteq A$ for some m in M. Clearly in Rm there exists m' in M. But, since $(yRm)\supseteq(ym', yRm')=(y)(m')$, $(y)(m')\subseteq A$. Thus $L'\subseteq L$.

From Lemma 2.7 (resp.([3], §8. Lemma 1) $l(A, M)$(resp. $u(A, M)$ coincides with $l_l(A, M)$(resp. $u_l(A, M)$ in [3]. Therefore from [3] we obtain the following lemma.

Lemma 2.8. *For any m-system M, $l(A,M)\subseteq u(A, M)$.*

Furthermore ([3], Theorem 12) holds in our case too as follows:

Theorem 2.4. *In R with (K), for any m-system M $u(A, M)=l(A, M)$.*

Proof. If $M\neq\phi$, by the definition $u(A, M)=l(A, M)=A$. If $M\cap A\neq\phi$, then $u(A, M)=l(A, M)=R$. And we shall show that, if $M\neq\phi$ and $M\cap A=\phi$, every element m in M is rp to $L=l(A, M)$.

Suppose that $(x)(m)\subseteq L$ for some m in M. From (K), $(x)(m)=(t_1,\cdots,t_n)$ where, for each i, $t_i\in L$, then there exists $m_i\in M$ such that $(t_i)(m_i)\subseteq A$. Since M is an m-system, there exists $m'=m_1x_1m_2\cdots x_{n-1}m_n\in M$ where every x_i is in R. Now, $(x)(m)(m')=(t_1,\cdots, t_n)(m')\subseteq(t_1,\cdots, t_n)(\bigcap_{i=1}^{n}(m_i))\subseteq A$. Of course, $(m)(m')\supseteq mRm'$, then there exists an element m″ in $(m)(m')\cap M$, and hence $(x)\cdot(y)(m'')\subseteq A$. Thus $x\in L$, and hence $u(A, M)\subseteq L$. The converse inclusion follows from Lemma 2.8.

From the proof of ([3], Theorem 13) we have the following theorem:

Theorem 2.5. *In R, for any m-system M $u(A\cap B, M)=u(A, M)\cap u(B, M)$.*

From now on, if $M=P^c$, where P^c is the complement of a prime ideal P, we shall write u(A,P) (resp. $l(A, P)$) for u$(A. M)$ (resp. $l(A, M)$).

We can prove the following lemma and corollaries in a way similar to ([3], Theorem 14, Corollary 2 and Corollary 3).

Lemma 2.9. *Let A be an ideal of R with* (F) *and* (K), *then every associated prime ideal of A is nrp to A and every ideal that is nrp to A is contained in one of the associated primes of A.*

Corollary 1. *In R with* (F) *and* (K), *maximal nrp to A ideal coincides with one of the associated primes of A.*

Corollary 2. *Let P be a prime ideal of R with* (F) *and* (K), *then $A=u(A, P)$ if and only if every associated prime of A is contained in P.*

Corollary 3. *In R with* (F) *and* (K), *for any ideal A and for any its prime divisor P, $u(A, P)$ is the intersection of all ideals containing A whose all associated prime ideals are contained in P.*

Theorem 2.6. *In R with* (F) *and* (K) *if A is a primal ideal, then A^* is a prime ideal.*

Proof. Let $A=T_1\cap\cdots\cap T_n$ be a normal tertiary decomposition of A, then from Lemma 2.9, Corollary 1, we easily obtain that $A^*=ter(T_i)$ for some i.

In [5], W. E. Barnes, expanding the concept of the upper M-component, has defined the upper B-component for any ideal B containig A. That is, u(A, B) *is defined as $u(A, M)$ where M is the set of elements that are rp to B.* Of course, M is not always an m-system.

Lemma 2.10. *Let A and B be any ideals of R with* (F) *and* (K). *Then $A=u(A, B)$ if and only if every associated prime of A is contained in an associated prime ideal of B. Further, $u(A, B)$ is the intersection of all ideals containing A each of whose associated prime ideals is contained in an associated prime ideal of B.*

Proof. Let M be as stated above. Now, $A=u(A, B)\Longleftrightarrow A=u(A, M)\Longleftrightarrow$each element m in M is rp to $A\Longleftrightarrow$ each element nrp to A is nrp to $B\Longleftrightarrow$ each associated prime ideal of A is contained in an associated prime ideal of B. Further, $u(A, B)=u(A, M)=\cap\{C$ $C\supseteq A$ and each element m in M is rp to $C\}=\cap\{D$ $D\supseteq A$ and each element x nrp to D is nrp to $B\}=\cap\{E$ $E\supseteq A$ and each associated prime ideal of E is contained in an associated prime ideal of $B\}$.

Theorem 2.7. *In R with* (F) *and* (K), *let B be an ideal with associated primes $P_1,\cdots, P_n$, then*

$u(A, B)=\bigcap_{i=1}^{n} u(A, P_i)$.

Proof. The proof is exactly the same as in the case of ([3], Theorem 15).

Theorem 2.8. *In R with* (F) *and* (K), *let P be an associated prime of A, then P is nrp to $u(A, P)$. Further $u(A, P)$ is a P-primal.*

Proof. This is also proved in the same way as ([3], Theorem 16).

Corollary. *In R with* (F) *and* (K), *let $P_1, \cdots, P_n$ be the maximal nrp to A ideals, then*

$A=u(A, P_1) \cap \cdots \cap u(A, P_n)$

is a primal decomposition of A.

Proof. From Lemma 2.9. Corollary 1, every P_i is prime. The rest is proved in the same way as ([3], Theorem 16. Corollary).

Following ([3], Theorem 17), we obtain the following theorem.

Theorem 2.9. *In R with* (F) *and* (K), *let P be a minimal prime divisor of A, then $u(A, P)$ is a P-tertiary ideal.*

Proof. Let P be a minimal prime divisor of A and let

(2.5) $u(A, P) = T_1 \cap \cdots \cap T_n$

be a normal tertiary decomposition where ter $(T_i)=P_i$ for eaeh i. From Lemma 2.9, each P_i is nrp to $u(A, P)$. On the other hand, any element that is not contained in P is rp to $u(A, P)$ from Lemma 2.6. Hence for each i $P_i \subseteq P$. Since P_i is a prime divisor of A, $P_i=P$ for each i. Therefore from (2.5) $u(A,P)$ is P-tertiary.

Lemma 2.11. *In R with (F) and* (K), *let*

$A=Q_1 \cap \cdots \cap Q_n$

be an irredundant primary decomposition. Then an element a is rp to A if and only if a is contained in each $\bar{Q}_i^c$.

Proof. Each primary Q_i is tertiary. Hence for each i, $\bar{Q}_i=Q_i^*=ter(Q_i)$ is an associated prime divisor of A. Thus from Lemma 2.9 the proof is completed.

Theorem 2.10. *In with* (F) *and* (K), *let*

$A=Q_1 \cap \cdots \cap Q_n$

be an irredundant primary decomposition where $\bar{Q}_i=P_i$, then the minimal prime divisors of A are exactly those primes which are minimal in the set $P_1, \cdots, P_n$.

Proof. Let P be a minimal prime divisor of A, then P contains at least one of those $P_1, \cdots, P_n$. In fact, from the assumption $\bar{A}=\bar{Q}_1 \cap \cdots \cap \bar{Q}_n = P_1 \cap \cdots \cap P_n \subseteq P$, hence $P_1 P_2 \cdots P_n \subseteq P$. Thus $P \supseteq P_k$ for some k. Since P is a minimal prime divisor of A, $P=P_k$. Hence the converse is obvious too.

Corollary. *Let R be a ring with* (F), (K) *and the Noetherian ideal theory, then any ideal has at most a finite number of minimal prime divisors.*

Lemma 2.12. *In R with* (F) *and* (K), *let*

$A=Q_1 \cap \cdots \cap Q_n$

be a primary decomposition of A where $P_i=\bar{Q}_i$ for each i. If P is a prime ideal which contains $P_1, \cdots, P_r$ but does not contain $P_{r+1}, \cdots, P_n$, then $u(A, P) = Q_1 \cap \cdots \cap Q_r$.

Proof. From Lemma 2.10, $u(A, P) \subseteq Q_1 \cap \cdots \cap Q_r$. Conversely, for each $j>r$, $P \not\supseteq Q_j$. In fact, if $P \supseteq Q_j$, then $P \supseteq P_j$, This is a contradiction. Hence for each i ($1 \leq i \leq n-r$), there exists an element m_i in Q_{r+i} such that $m_i \notin P$. Since P^c, the complement of P, is an m-system, there exists $m=m_1 r_1 m_2 \cdots r_{n-r-1} m_{n-r} \in P^c$ where $r_i \in R$. Hence $m \in Q_{r+1} \cap \cdots \cap Q_n$. Therefore, for any element $q \in Q_1 \cap \cdots \cap Q_r$, $(q)(m) \in Q_1 \cap \cdots \cap Q_n=A$. Thus $q \in l(A, P)$. It follows that $u(A, P)=Q_1 \cap \cdots \cap Q_r$.

Theorem 2.11. *In R with (F) and (K), let*

(2.6) $A=Q_1\cap\cdots\cap Q_n$

be a normal primary decomposition of A. Then for any prime ideal $P\neq R$ which contains A, $P=\bar{Q}_k$ for some k if and only if P is nrp to $u(A,P)$.

Proof. For each i, let $\bar{Q}_i=P_i$. As to "*only if part*", from Theorem 2.3 P is an associated prime ideal of A. Therefore, from Theorem 2.8, P is nrp to $u(A,P)$. As to "*if part*", it is proved in a way similar to the proof of ([2], Theorem 18).

Now we shall define the following condition weaker than the Noetherian ideal theory.

(H). *For any ideal A, if ter(A) is prime, then ter(A) is a minimal prime divisor of A.*

Theorem 2.12. *Let R be a ring with* (H), (F) *and* (K), *and let*

(2.8) $A=T_1\cap\cdots\cap T_n$

be a normal tertiary decomposition of A where $P_i=ter(T_i)$ for each i, then for any set of ideals $X_1,\cdots,X_n$ satisfying $A\subseteq X_i\subseteq T_i$ and $ter(X_i)=P_i$ for each i,

(2.9) $A=u(X_1,P_1)\cap\cdots\cap u(X_n,P_n)$

is a normal tertiary decomposition of A.

Proof. From $A\subseteq X_i\subseteq T_i\subseteq P_i$ and from Lemma 2.9. Corollary 2, $A\subseteq u(X_i,P_i)\subseteq u(T_i,P_i)=T_i$. Since $ter(X_i)=P_i$, it follows from (H) that P_i is a minimal prime divisor of X_i for each i. Then from Theorem 2.9, $u(X_i,P_i)$ is P_i-tertiary. Now, $A\subseteq u(X_1,P_1)\cap\cdots\cap u(X_n,P_n)\subseteq T_1\cap\cdots\cap T_n=A$, hence (2.9) follows immediately. An irredundancy of (2.9) is obtained from Theorem 2.2.

Corollary. *Let R be a ring with* (F),(K) *and the Noetherian ideal theory and let*

(2.10) $A=Q_1\cap\cdots\cap Q_n$

be a normal primary decomposition of A where $P_i=\bar{Q}_i$ for each i. Then for any set of ideals $X_1,\cdots,X_n$ satisfying $A\subseteq X_i\subseteq Q_i$ and $\bar{X}_i=P_i$ for each i,

$A=u(X_1,P_1)\cap\cdots\cap u(X_n,P_n)$

is a normal primary decomposition of A.

Remark. In above corollary, for each i let $X_i=(P_i^{n_i}\cap Q_i)+A$ for any positive integer n_i, then X_i satisfies that $A\subseteq X_i\subseteq Q_i$ and $\bar{X}_i=P_i$.

Definition 2.6. An ideal A is called *a quasi P-primary* if $\bar{A}$ is a prime ideal P.

Theorem 2.13. *Let R be a ring with* (F), (K) *and the Noetherian ideal theory and let $P_1,P_2,\cdots,P_r$ be minimal prime divisors of A. Then there exists the set of ideals $X_1,X_2,\cdots,X_r$ such that, for each i, $\bar{X}_i=P_i$ and such that*

(2.10) $A=X_1\cap\cdots\cap X_r$

is an irredundant quasi primary decomposition of A.

Proof. Let $A=Q_1\cap\cdots\cap Q_n$ be a normal primary decomposition of A where $\bar{Q}_i=P_i$ for each i, and let $P_j\supseteq P_{i(j)}$ $(r<j\leqq n,\ 1\leqq i(j)\leqq r)$. If $1=i(j_1)=\cdots=i(j_t)$, then we shall set $X_1=Q_1\cap Q_{j_1}\cap\cdots\cap Q_{j_t}$, and then $X_1\subseteq Q_1$ and $X_1\subseteq Q_{j_1},\cdots,X_1\subseteq Q_{j_t}$, hence $\bar{X}_1=\bar{Q}_1\cap\bar{Q}_{j_1}\cap\cdots\cap\bar{Q}_{j_t}=P_1\cap P_{j_1}\cap\cdots\cap P_{j_t}=P_1$, and so on. Now we have $A=Q_1\cap\cdots\cap Q_n\supseteq X_1\cap\cdots\cap X_r\cap Q_{r+1}\cap\cdots\cap Q_n\supseteq X_1\cap\cdots\cap X_r\supseteq A$. Hence we obtain (2.10). If $X_1\supseteq X_2\cap\cdots\cap X_r$, then $\bar{X}_1\supseteq\bar{X}_2\cap\cdots\cap\bar{X}_r$, hence $P_1\supseteq P_2\cap\cdots\cap P_r\supseteq P_2P_3\cdots P_r$. This is contradictory to the fact that P_1 is a minimal prime divisor of A.

Theorem 2.14. *Let R be a ring with* (F), (K) *and the Noetherian ideal theory, and let P be a prime divisor of A. Then the following statements are equivalent to one another.*

(1). *P is a minimal prime divisor of A.*

(2). *$u(A,P)$ is P-primary.*

(3). *There exists an ideal X such that* $A \subseteq X$, $\overline{X}=P$ *and* $u(A, P)=u(X, P)$.

(4). *There exists an ideal X such that* $\overline{X}=P$ *and* $X \subseteq u(A, P)$.

Further each of the statements (1)——(4) *implies*

(5). $u(A, P)$ *is a minimal primary divisor of A.*

Proof. (1)$\Longrightarrow$(3) : From Theorem 2.10 we may assume that $P=P_1$ in that theorem. Then in Theorem 2.13 $u(A, P)=u(A, P_1)=u(X_1, P_1) \cap \cdots \cap u(X_r, P_1)$, where $X_i \supseteq A$ for each i and $\overline{X}_1, \cdots, \overline{X}_r$ are exactly minimal prime divisors of A. And then for $k \neq 1$, $X_k \not\subseteq P_1$. In fact, if $X_k \subseteq P_1$, then $\overline{X}_k \subseteq P_1$, and this is a contradiction. Hence $u(X_k, P_1)=R$. Therefore $u(A, P)=u(X_1, P)$ where $A \subseteq X_1$ and $\overline{X}_1=P$.

(3)$\Longrightarrow$(4) : This is evident.

(4)$\Longrightarrow$(3) : For X in (4) let $Y=A+X$, then it is clear that $Y \supseteq A$ and $\overline{Y}=P$. Hence $A \subseteq Y \subseteq P$ and $u(A, P) \subseteq u(Y, P)$. On the other hand, from (4) $Y \subseteq u(A, P)$, then $u(Y, P) \subseteq u(u(A, P), P)=u(A, P)$ from Theorem 2.4.

(3)$\Longrightarrow$(2) : Since P is a minimal prime divisor of X, $u(X, P)$ is P-tertiary; consequently it is P-primary.

(2)$\Longrightarrow$(1) : If there exists a prime divisor P' such that $A \subseteq P' \subset P$, then $u(A, P) \subseteq u(A, P') \subseteq P'$. *Hence* $\overline{u(A, P)} \subseteq P'$. This is contradictory to $\overline{u(A, P)}=P$.

(1)$\Longrightarrow$(5) : Let Q be a primary ideal such that $u(A, P) \supseteq Q \supseteq A$, then $P \supseteq Q$, and $u(A, P) \subseteq Q$. Hence $u(A, P)=Q$.

§ **3.** We shall define the following condition :

(Q). *Every ideal of R is represented as an intersection of a finite number of quasi primary ideals.*

It is clear that R *with (F)* has (Q) if and only if every $\cap$-irreducible ideal is a quasi primary ideal.

Theorem 3.1. *In R with* (F) *and* (K), *the following statements are equivalent to one another.*

(1). *R has the Noeherian ideal theory.*

(2). *in Theorem* 1.1.

(3). *in Theorem* 1.1.

(4). *in Theorem* 1.1.

(5). *in Theorem* 1.1.

(6). *in Theorem* 1.1.

(7). *For any ideals A and B, there exists an ideal B' such that* $\overline{B}=\overline{B'}$ *and* $A \cap B' \subseteq AB$.

(8). *For any ideal A, any prime ideal P which is nrp to* $u(A, p)$ *occurs as an associated prime ideal of A.*

(9). *For any ideal A, any minimal prime divisor of A occurs as an asssociated prime ideal of A.*

(10). *For any ideals A and B such that* $A \subseteq B$, $ter(A) \subseteq ter(B)$.

(11). *R has* (H), *and for any ideal A and for any its minimal prime divisor P*, $u(A, P)$ *is primary.*

(12). *R has* (H) *and* (Q).

(13). *For any ideal A, let*

(3.1) $A=T_1 \cap \cdots \cap T_n$

be any normal tertiary decomposition of A where $P_i=ter(T_i)$ *for each i. If a prime ideal P contains* $P_1, \cdots, P_r$ *and if it does not contain* $P_{r+1}, \cdots, P_n$, *then* $u(A, P)=T_1 \cap \cdots \cap T_r$.

(14). *Let the condition be the same as in* (13). *If P is a minimal prime divisor of A, then* $u(A, P)=T_i$ *for some i.*

(15). *For any ideal A and for any its minimal prime divisor* P_1, *there exist prime divisors* $P_1, P_2, \cdots, P_n$ *of A and ideals* $X_1, X_2, \cdots, X_n$ *such that* $ter(X_i)=P_i$ *for each i, and such that*

(3.2) $A=u(X_1, P_1) \cap u(X_2, P_2) \cap \cdots \cap u(X_n, P_n)$

is a normal tertiary decomposition of A.

(16). *For any ideal A and for any its minimal prime divisor* P_1, *there exist ideals* $X_1, X_2, \cdots, X_r$ *such thet* $ter(X_1)=P_1$ *and such that* $A=X_1 \cap X_2 \cap \cdots \cap X_r$ *is an irredundant decomposition.*

Proof. It is obtained from Theorem 1.1 and Theorem 1.2 that the statements (1)—(7) are equivalent to one another.

(1)$\Longrightarrow$(13) : Let (3.1) be any normal tertiary decomposition where $P_i=ter(T_i)$ for each i. From (1), it is a normal primary decomposition, too; consequently we also have, from Lemma 2.12, that $u(A, P)=T_1 \cap \cdots \cap T_r$.

(13)$\Longrightarrow$(8) : Let a prime ideal P be nrp to $u(A, P)$, then $u(A, P) \neq R$, hence $A \subseteq P$. In (3.1) if P contains $P_1, \cdots, P_r$ but if it does not contain $P_{r+1}, \cdots, P_n$, then, from (13) we have

(3.3) $u(A, P)=T_1 \cap \cdots \cap T_r$.

It is clear that (3.3) is a normal tertiary decomposition, and therefore from Lemma 2.9 $P \subseteq P_i$ for some i $(1 \leqq i \leqq r)$. But since $P \supseteq P_i$ we have $P=P_i$.

(8)$\Longrightarrow$(9) : From Theorem 2.9, this is evident.

(9)$\Longrightarrow$(4) : For any ideal A, $\bar{A}=\cap\{C \mid C$ is a minimal prime divisor of $A\} \supseteq \cap\{D \mid D$ is an associated prime divisor of $A\}=ter(A)$.

(1)$\Longrightarrow$(14) : From (9) a minimal prime divisor P of A is a minimal associated prime ideal of A. Then in (3.1) P contains only some P_i, hence from (13) $u(A, P)=T_i$.

(14)$\Longrightarrow$(5) : Let A be a tertiary ideal and let P be a minimal prime divisor of A, then we have, from (14), $u(A, P)=A$. Therefore from Lemma 2.9 Corollary 2, $ter(A)=P$, hence $ter(A)=\bar{A}$. Thus a tertiary ideal is a primary.

(1)$\Longrightarrow$(10) : For any ideal A $ter(A)=\bar{A}$, and we obtain (10).

(10)$\Longrightarrow$(5) : Let A be a tertiary ideal and let P be a minimal prime divisor of A. Since P is primary, P is tertiary. Then $P=P^*=ter(P)$. As $A \subseteq P$, from (10) $ter(A) \subseteq ter(P)=P$. Thus $ter(A) \subseteq \bar{A}$.

(1)$\Longrightarrow$(11) : This is evident from (4) and Theorom 2.14.

(11)$\Longrightarrow$(5) : For any tertiary ideal A, $ter(A)=P$ is a minimal prime divisor of A from (H). Hence $u(A, P)$ is primary. On the other hand, from Lemma 2.9. Corollary 2, $u(A, P)=A$. Thus we obtain (6).

(1)$\Longrightarrow$(12) : This is evident from (4).

(12)$\Longrightarrow$(2) : For any $\cap$-irreducible ideal A, from (Q) $\bar{A}$ is a prime P and from (H) $ter(A)=P$. Hence A is primary.

(1)$\Longrightarrow$(15) : This is evident from Theorem 2.12. Corollary, its Remark and (9).

(15)$\Longrightarrow$(5) : For any tertiary ideal A and for any its minimal prime divisor P, from (15) there exists an ideal X such that $A=u(X, P)$ and $ter(X)=P$. We shall show that $ter(u(X, P))=P$. Let $X=T_1 \cap \cdots \cap T_s$ be a normal tertiary decomposition, then $P=ter(X)=ter(T_1) \cap \cdots \cap ter(T_s)$. Since a prime ideal is $\cap$-irreducible, $P=ter(T_k)$ for some k. Hence from Theorem 2.8, $u(X,P)$ is P-primal. And since $u(X, P)$ is tertiary, $ter(u(X, P))=u(X, P)^*=P$. Thus $ter(A)=P$. Since P is any minimal prime divisor of A, $ter(A)=\bar{A}$.

(1)$\Longrightarrow$(16) : This is evident from Throrem 2.13.

(16)$\Longrightarrow$(2) : For any $\cap$-irreducible ideal A and for any its minimal prime divisor P, from (16) there exists an ideal X such that $ter(X)=P$ and such that $A=X$. Hence $ter(A)=P$. Since P is any minimal prime divisor of A, ter $(A)=\bar{A}$.

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