On CETS-Modules in a torsion theory I

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Abstract

Patrik F . Smith [3] defined the CESS-modudes and obtained several basic results on these modules. In this paper , we generalize the CESS-modules in a torsion theory.

Let t be a left exact preradical with following property. *If N is an essential submodule of* M, *then* $t(N)=t(M)$. Using this preradical, we show the following which is our main result : For a module $M=M_1\oplus M_2\oplus\cdots\oplus M_n$, M is CETS if and only if every closure K of a torsion submodule of M with $K\cap M_i=0$ for some $1\leq i\leq n$, is a direct summand of M.

1. Preliminaries

Throughout this paper, all rings are associative with identity and all modules are unital right modules. Let R be a ring and M an R-module. A submodule K of M is called closed in M if K has no proper essential extension in M. By Zorn's Lemma, for every submodule N of M, there exists a closed submodule K of M such that N is essential in K, and in this case we call K a closure of N in M. Again, let M be any module ,and let L be any submodule of M. By Zorn's lemma, the collection of submodules H of M such that $H\cap L=0$ has a maximal member P. P is called a *complement* of L (in M). A submodule K of M is called a complement submodule if there exists a submodule Q of M such that K is a complement of Q in M. It is well known that a submodule K of M is closed if and only if K is a complement.

The module M is called a CS*-module* if every complement submodule is a direct summand. CS-modules are often called *extendig* modules by some authors. It is clear that a module is a CS-module if and only if every submodule is essential in a direct summand.

Let N be a submodule of M. $N\leq_e M$ and $N\leq_{cl} M$ mean that N is essential in M and N is closed in M, respectively.

For each preradical t, we denote the t *-torsion (resp.* t *-torsionfree)* class by T(t) (resp.F(t)).

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For all undefined notions about torsion theories the reader is referred to Golan [2] and Stenström [4] .

Now, let t be a left exact preradical with the following property. *If N is an essential submodule of M, then $t(N)=t(M)$,where $t(M)$ means the torsion submodule of M.*

2. CETS-modules

Followig [3] , a module M is called a CESS-module if every complement with essential socle is a direct summand, equivalently, every submodule with essential socle is essential in a direct summand of M.

A module M is called a *CETS-module* if every complement N with $t(N) \leq_e N$ is a direct summand, that is, every submodule N with $t(N) \leq_e N$ is essential in a direct summand of M, or more equivalently, evey closure of any torsion submodule is a direct summand of M.

Remark 1. CS-modules are CETS-modules.

Remark 2. For the preradical $t = socle$, CETS-modules are the same to CESS-modules.

Remark 3. Torsion free modules are CETS.

Remark 4. Torsion modules are semisimple modules. So these modules are CS - and CETS-modules.

Lemma 1. Let M be a CETS-module with $t(M) \leq_e M$. Then, M is a CS-module.

Proof. Let N be any complement in M. We have that $t(N) \leq_e N$. Since M is CETS, we see that N is a direct summand of M. Hence, M is CS.

Lemma 2. Any direct summand of CETS-modules is a CETS-module.

Proof. Let M be CETS and let $M = M_1 \oplus M_2$. Let N is a closed submodule of M_1 with $t(N) \leq_e N$. We see that N is closed in M. So, there is a submodule X of M such that $M = N \oplus X$. Then we have $M_1 = N \oplus (M_1 \cap X)$, so N is a direct summand of M_1. Hence M_1 is CETS.

Lemma 3. A module M is CETS if and only if every closure of the $t(M)$ is CS and a direct summand of M.

Proof. Suppose first that M is a CETS-module. Now, let $\overline{t(M)}$ be any closure of

$t(M)$. We have that $t(\overline{t(M)}) \leq_e \overline{t(M)}$. Then $\overline{t(M)}$ is a direct summand of M, because M is CETS. By **Lemma 2**, $\overline{t(M)}$ is CETS and by **Lemma 1**, $\overline{t(M)}$ is CS. Conversely, let N be a complement submodule of M with $t(N) \leq_e N$. By the assumption for preradical t, we obtain that $t(M) = t(N) \oplus L \leq_e N \oplus L$ for some submodule L of $t(M)$. Let K be closure of $N \oplus L$ in M. Then K is closure of $t(M)$. By the assumption, K is CS and K is a direct summand of M. Since N is complement in K and N is a direct summand of M, it follows that M is CETS.

Corollary 4. Let $M = M_1 \oplus M_2$ where M_1 is a torsion submodule and M_2 is a torsion free submodule. Then M is a CETS-module.

Proof. Clearly $M_1 = t(M)$ and hence M_1 is closure of $t(M)$. By **Remark 4** and **Lemma 3**, M is CETS.

Remark 5 For our preradical t, if t is *splitting* then every module is CETS.

Corollary 5. Let M be an R-module such that $t(M) \leq_e M$. Then, M is CS if and only if M is CETS.

Proof. Only if part is clear. If part is follows from **Lemma 1**.

Proposition 6. Let $M_i (1 \leq i \leq n)$ be a finite collection of R-modules and let $M = M_1 \oplus \cdots \oplus M_n$. Then M is CETS if and only if every closure K of a torsion submodule of M with $K \cap M_i = 0$ for some $1 \leq i \leq n$, is a direct summand of M.

Proof. The necessity is clear. Conversely, suppose that M has the stated condition. Let K be a closed submodule of M with $t(K) \leq_e K$. Let $M' = M_1 \oplus \cdots \oplus M_{n-1}$. Let H be a closure in K of $K \cap M'$. Note that H is closed in M and H has essential torsion part. (i.e. $t(H) \leq_e H$) Since $H \cap M_n = 0$ and H is a closure of $t(K \cap M')$ in M, by hypothesis, H is a direct summand of M. So, there exists a submodule H' of M such that $M = H \oplus H'$. Then, $K = H \oplus (K \cap H')$. We see that $K \cap H'$ is closed in M, $t(K \cap H')$ is essential in $K \cap H'$ and $(K \cap H') \cap M_1 = 0$. By hypothesis, $K \cap H'$ is a direct summand of M and hence also of H'. It follows that K is a direct summand of M. Thus, M is CETS.

Given a finite collection of modules M_i $(1 \leq i \leq n)$, we say that the modules are *relatively injective* if M_i is M_j-injective for all $i \neq j$ in $\{1,2 \cdots, n\}$.

Corollary 7. Let M_i $(1 \le i \le n)$ be a finite collection of relatively injective R-modules. Then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is CETS if and only if M_i is CETS for each i $(1 \le i \le n)$.

Proof. The necessity is clear by **Lemma 2**. Conversely, suppose that each $M_i (1 \le i \le n)$ is CETS. By induction on n, we can suppose without loss of generality that $n = 2$. Let K be a closed submodule of $M = M_1 \oplus M_2$ with $t(K) \le_e K$. Suppose that $K \cap M_1 = 0$. It is well known that there exists a submodule M' of M such that $M = M \oplus M'$ and $K \le M'$. Clearly $M' \cong M_2$, so that M' is CETS. Hence K is a direct summand of M', and hence also of M . Similarly, if L is a closed submodule with $t(L) \le_e L$ and with $L \cap M_2 = 0$, then L is direct summand of M. Moreover ,K and L are closure of $t(K)$ and $t(L)$, respectively. So, by **Proposition 6** , M is CETS.

Proposition 8. Let M be a CETS module. Then M has a decomposition $M = M_1 \oplus M_2$ such that M_1 is CS, $t(M_1) \le_e M_1$ and $t(M_2) = 0$.

Proof. Since M is CETS, there is a direct summand M_1 of M such that $t(M) \le_e M_1$. We see from **Lemma 1** that M_1 is CS. Now , let $M = M_1 \oplus M_2$. Then, clearly, $t(M_2) = 0$.

References

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