Induced Velocity of Propellers at Zero Advanced Ratio in Consideration of Slip Stream Contraction

by M. Takeshige*

1. Summary

The formula for the calculation of propellers has been introduced by M. Iwasaki, S. Kawada T, Mo riya. Applying these formula in the calculation, the cylindrical voltex sheets have constant radius without contraction.

Practically visualisation of the flow through a model roter by means of smoke has shown the contrac tion of the herical wake, so the author tried to lead the formula in consideration of slipstream contracti-on, using Moriya's mathematical expression for the herical wake

2. List of Symbols

φ : parametric variable of a helix
N : number of blades
V : velocity of oncoming flow
Ω : angular velocity of propeller
h : V/Ω
Γ : circulation round a blade
r : distance of any blade element from axis of propeller
r' : distance of any blade element from the axis of propeller which is located on the y-axis
R : tip radius of propeller
R : distance between the point P on the helical vortex filament and the point Q which is located on the blade passing through the y-axis
ds : length of vortex element
ν : angle measured between tanzential line at the point P and the line PQ
λ : corresponding speed ratio of the propeller

3. Co-ordinate System used in the Description of Herical wake.

For a description of the helical wake, a non-rotating co-ordinate system fixed in space. The cartesian co-ordinates x, y, z or cylindrical polar co-ordinates r, φ, z will be used with the usual relationships between them.

if the position of P is stated in cylindrical co-ordinates $P(x', y', z')$, its cylindrical co-ordinates are given by (See Fig. 1)

$$x'=h\varphi, \quad y'=re^{-p\varphi}\cos\varphi_k \qquad z'=re^{-p\varphi}\sin\varphi_k$$

with r positive and φ is parameter

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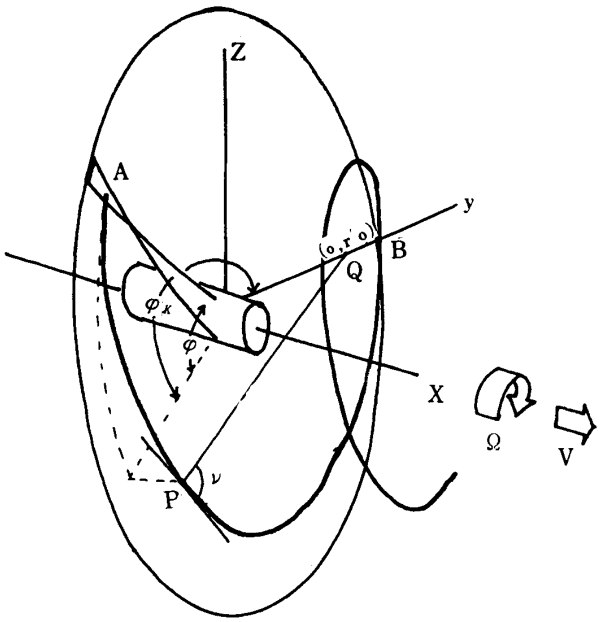

Fig. 1 the geometrical relations of the wake spirl

4. Basic Equation

In the Biot-Savart law, the following expression is obtained for the induced velocity dw at the point Q due to an element of length ds on the wake helix of strength $d\Gamma$ at the point P

$$dw=-\frac{d\Gamma}{4\pi}\frac{\sin\nu}{\bar{R}^2}ds$$

where $\bar{R}=\overline{PQ}=\sqrt{h^2\varphi^2+(re^{-p\varphi}\cos\varphi_k-r')^2+r^2e^{-2p\varphi}\sin^2\varphi_k}$

$$=\sqrt{r^2e^{-2p\varphi}-2rr'e^{-p\varphi}\cos\varphi_k+r'^2+h^2\varphi^2}$$

The equation of the line $\overline{PQ}$ can be written as

$$\frac{x}{h\varphi}=\frac{y-r'}{re^{-p\varphi}\cos\varphi_k-r'}=\frac{z}{re^{-p\varphi}\sin\varphi_k}$$

and the equation of tangent of the herical wake at the point p can therefore be written as

$$\frac{x-h\varphi}{h}=\frac{y-re^{-p\varphi}\cos\varphi_k}{-rpe^{-p\varphi}\sin\varphi_k}=\frac{z-re^{-p\varphi}\sin\varphi_k}{rpe^{-p\varphi}\cos\varphi_k}$$

where $\sin\nu=\dfrac{\sqrt{\begin{vmatrix}re^{-p\varphi}\cos\varphi_k-r' & re^{-p\varphi}\sin\varphi_k\\ -rpe^{-p\varphi}\sin\varphi_k & rpe^{-p\varphi}\cos\varphi_k\end{vmatrix}^2+\begin{vmatrix}re^{-p\varphi}\sin\varphi_k & h\varphi\\ rpe^{-p\varphi}\cos\varphi_k & h\end{vmatrix}^2+\begin{vmatrix}h\varphi & re^{-p\varphi}\cos\varphi_k-r'\\ h & -rpe^{-p\varphi}\sin\varphi_k\end{vmatrix}^2}}{\bar{R}\cdot\sqrt{h^2+r^2p^2e^{-2p\varphi}}}$

and $ds=\sqrt{(dx')^2+(dy')^2+(dz')^2}=\sqrt{h^2+r^2p^2e^{-2p\varphi}}\,d\varphi$

The indnced velocity is perpendicular to R and ds.

If directions of cosine are given as l, m, and n, we oblain

$$lh\varphi+m(re^{-p\varphi}\cos\varphi_k-r')+nre^{-p\varphi}\sin\varphi_k=0$$

$$lh+m(-rpe^{-p\varphi}\cos\varphi_k)+npre^{-p\varphi}\sin\varphi_k=0$$

Let $A=\sqrt{\begin{vmatrix}re^{-p\varphi}\cos\varphi_k-r' & re^{-p\varphi}\sin\varphi_k\\ -rpe^{-p\varphi}\sin\varphi_k & rpe^{-p\varphi}\cos\varphi_k\end{vmatrix}^2+\begin{vmatrix}re^{-p\varphi}\sin\varphi_k & h\varphi\\ rpe^{-p\varphi}\cos\varphi_k & h\end{vmatrix}^2+\begin{vmatrix}h\varphi & re^{-p\varphi}\cos\varphi_k-r'\\ h & -rpe^{-p\varphi}\sin\varphi_k\end{vmatrix}^2}$

then direction of cosine becomes respectively

$$l=\frac{1}{A}\begin{vmatrix}re^{-p\varphi}\cos\varphi_k-r' & re^{-p\varphi}\sin\varphi_k\\ -rpe^{-p\varphi}\sin\varphi_k & rpe^{-p\varphi}\cos\varphi_k\end{vmatrix}$$

$$m=\frac{1}{A}\begin{vmatrix} re^{-p\varphi}\sin\varphi_k & h\varphi \\ rpe^{-p\varphi}\cos\varphi_k & h \end{vmatrix}$$

$$n=\frac{1}{A}\begin{vmatrix} h\varphi & re^{-p\varphi}\cos\varphi_k-r' \\ h & rpe^{-p\varphi}\sin\varphi_k \end{vmatrix}$$

Here the element of indnced velocity in the x-, y-, and z-directions at the point Q are given by.

$$dw_x=\frac{1}{A}\begin{vmatrix} re^{-p\varphi}\cos\varphi_k-r' & re^{-p\varphi}\sin\varphi_k \\ -rpe^{-p\varphi}\sin\varphi_k & rpe^{-p\varphi}\cos\varphi_k \end{vmatrix}dw$$

$$dw_y=\frac{1}{A}\begin{vmatrix} re^{-p\varphi}\sin\varphi_k & h\varphi \\ rpe^{-p\varphi}\cos\varphi_k & h \end{vmatrix}dw$$

$$dw_z=\frac{1}{A}\begin{vmatrix} h\varphi & re^{-p\varphi}\cos\varphi_k-r' \\ h & rpe^{-p\varphi}\sin\varphi_k \end{vmatrix}dw$$

or $dw_x=ldw \qquad dw_y=mdw \qquad dw_z=ndw$

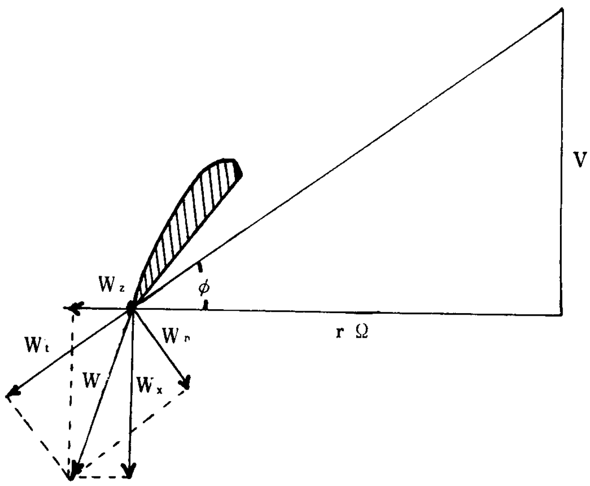

Fig. 2 the speed diagram for the blade element at radius r'.

so that the induced velocity which is normal to the resultant speed $\sqrt{V^2+(r'\Omega)^2}$ at the point $Q(o,\ r',\ o)$ becomes $\quad dw_n=dw_x\cos\phi-dw_z\sin\phi$

$$=\frac{\Omega r'}{\sqrt{V^2+\Omega^2 r'^2}}ldw-\frac{V}{\sqrt{V^2+\Omega^2 r'^2}}ndw$$

substituting Biot-Savart raw

$$=-\frac{d\Gamma}{4\pi}\ \frac{\sin\nu}{R^2}ds\ \frac{\Omega r'\ l-Vn}{\sqrt{V^2+\Omega^2\ r'^2}}$$

using equation $\sin\nu$, ds, l and n.

$$dw_n=-\frac{d\Gamma}{4\pi}\ \frac{d\varphi}{R^3}\ \frac{\Omega r r' e^{-p\varphi}\begin{vmatrix} re^{-p\varphi}\cos\varphi_k-r' & \sin\varphi_k \\ -rpe^{-p\varphi}\sin\varphi_k & p\cos\varphi_k \end{vmatrix}-Vh\begin{vmatrix} \varphi & re^{-p\varphi}\cos\varphi_k-r' \\ 1 & -rpe^{-p\varphi}\sin\varphi_k \end{vmatrix}}{\sqrt{V^2+(r'\Omega)^2}}$$

$$=-\frac{d\Gamma}{4\pi}d\varphi\frac{\Omega\left[rr'\ pe^{-p\varphi}(re^{-p\varphi}-r'\cos\varphi_k)+\frac{V}{\Omega}h\{re^{-p\varphi}\ (p\varphi\sin\varphi_k+\cos\varphi_k)-r'\}\right]}{\sqrt{r^2e^{-p\varphi}-2rr'e^{-p\varphi}\cos\varphi_k+r'^2+h^2\varphi^2}^{\,3}\sqrt{V^2+(r'\Omega)^2}}$$

$$=-\frac{d\Gamma}{4\pi R}d\varphi\frac{\frac{r}{R}\cdot\frac{r'}{R}pe^{-p\varphi}\left(\frac{r}{R}e^{-p\varphi}-\frac{r'}{R}\cos\varphi_k\right)+\left(\frac{V}{\Omega R}\right)^2\left\{\frac{r}{R}e^{-p\varphi}(p\varphi\sin\varphi_k+\cos\varphi_k)-\frac{r'}{R}\right\}}{\sqrt{\left(\frac{r}{R}\right)^2e^{-2p\varphi}-2\cdot\frac{r}{R}\cdot\frac{r'}{R}e^{-p\varphi}\cos\varphi_k+\left(\frac{r'}{R}\right)^2+\left(\frac{h}{R}\right)^2\varphi^2}^{\,3}\sqrt{\left(\frac{V}{R\Omega}\right)^2+\left(\frac{r'}{R}\right)^2}}$$

by introducing the nondimensional quantities

$$\xi=\frac{r}{R}, \quad \xi'=\frac{r'}{R} \text{ and } \lambda=\frac{V}{\Omega R}$$

$$=-\frac{d\Gamma}{4\pi R}\frac{\xi\cdot\xi' pe^{-p\varphi}(\xi\cdot e^{-p\varphi}-\xi'\cos\varphi_k)+\lambda^2\{\xi e^{-p\varphi}(p\varphi\sin\varphi_k+\cos\varphi_k)-\xi'\}}{\sqrt{\xi^2 e^{-2p\varphi}-2\xi\cdot\xi' e^{-p\varphi}\cos\varphi_k+\xi'^2+\lambda^2\varphi^2}^{\,3}\sqrt{\lambda^2+\xi'^2}}d\varphi$$

after all, the complete multiple integral for the contraction is obtained as

$$w_n(\xi')=\int_0^1-\frac{\frac{d\Gamma}{d\xi}d\xi}{4\pi R}\sum_{k=1}^{n}\int_0^\infty\frac{\xi\cdot\xi' pe^{-p\varphi}(\xi\cdot e^{-\varphi\varphi}-\xi'\cos\varphi_k)+\lambda^2\{\xi e^{-p\varphi}(p\varphi\sin\varphi_k+\cos\varphi_k)-\xi'\}}{\sqrt{\xi^2 e^{-2\varphi k}-2\xi\xi' e^{-p\varphi}\cos\varphi_k+\xi'^2+\lambda^2\varphi^2}\sqrt{\xi'^2+\lambda^2}}d\varphi$$

or $$w_n(\xi')=\int_0^1\frac{-\frac{d\Gamma}{d\xi}d\xi}{4\pi R}\cdot\frac{I}{\xi-\xi'}$$

The value of I is induction factor and represents a correctional factor.

5. Recommendations for Futnre Work

(1) The anthor shonld like to calculate the values of I and compare with the values which hane been clarified by T. moriya

(2) Experirnental stndies of tip volticity might be taken into consideration in mathematical model.

6. Acknowledgewent

The anthor wishes to express his gratitude to professor M. Iwasaki for his suggestions.

7. References

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3) M. Iwasaki : Calculation of Static thrnst of Propellers (Ⅰ) and (Ⅱ).

(昭和43年4月12日受理)