On the generalized t-full modules

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Abstruct

Let R be a ring with identity and t a left exact radical of R-mod. In this paper, we shall introduce the notion of the generalized t-full module and study its basic properties. And , it is shown that, for its corresponding Gabriel topology $\mathcal{L}(t)$ and a linear topology $\mathcal{L}=\{_R I \leqq R \mid I \geqq t(R)\}$, the linear topology $\mathcal{L}(t) \cap \mathcal{L}$ is contained in $\boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is the set of all essential left ideals of R. In particular, if R is generalized t-full as a left R-module, then $\mathcal{L}(t) \cap \mathcal{L}$ is just equal to the set $\boldsymbol{\varepsilon}$.

1. Preliminaries

Throughout this paper, R is a ring with identity and R-modules are unital left R-modules. R-mod. denotes the category of all R-modules. We denote by $E(M)$ the injective hull of $_R M$. Let t be a left exact radical of R-mod.

An R-module M is said to be a t-full module if it is torsionfree and for each essential submodule N of M, M/N is torsion (cf. [1]). Generalizing this concept, we define an R-module M to be generalized t-full (in short, $g.t$-full) *if N is an essential submodule of M, then M/N is torsion and $t(M)=t(N)$.*

Note that a t-full R-module is nothing but a torsionfree $g.t$-full R-module. As is easily seen $g.t$-full modules are not always t-full modules.

For all undefined notions about torsion theories we refer to Golan [1] , Kurata [2] and Stenström [3] .

2. Several properties of generalized t-full modules

As is well-known, if N is a (t-) dense submodule of a torsionfree R-module M, then N is essential in M. But, this property holds under a more weaker assumption as the following lemma shows :

Lemma 1. *Let M be an R-module and N a submodule of M. If M/N is torsion and $t(M)=t(N)$, then N is essential in M.*

Proof. First note that $t(M)=t(N)$ if and only if $N \geqq t(M)$. Now, since $M/t(M)$ is torsionfree and $(M/t(M))/(N/t(M))$ $(\cong M/N)$ is torsion, it follows that $N/t(M)$ is essential in $M/t(M)$. Hence N is essential in M.

From this lemma, we see that M $g.t$-full is equivalent to the following condition :

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for any submodule $N \leqq M$, $N \leqq_e M$ if and only if M/N is torsion and $t(M)=t(N)$.

Proposition 2. *If M is g.t-full and N is a submodule of M then N is also g.t-full.*

Proof. Let N' be an essential submodule of N and let M' be a complement of N in M. Then $N \oplus M'$ is essential in M and hence $N' \oplus M'$ is also essential in M. As M is $g.t$-full, $t(N' \oplus M')=t(M)$, $t(N \oplus M')=t(M)$ and $M/(N' \oplus M')$ is torsion. Thus $(N \oplus M')/(N' \oplus M')$ is torsion and so is N/N'. Moreover we see $t(N')=t(N)$. Thus N is $g.t$-full.

Proposition 3. *Let M be a g.t-full module and N a submodule of M with $N \leqq t(M)$. Then, M/N is g.t-full.*

Proof. Let N'/N be an essential submodule of M/N. Since N' is essential in M, $t(M)=t(N')$ and M/N' is torsion. Hence $(M/N)/(N'/N)$ is also torsion. Moreover, by the assumption, $t(M/N)=t(M)/N=t(N')/N=t(N'/N)$. Therefore, M/N is $g.t$-full.

Proposition 4. *Let M be an R-module and N a closed submodule of M such that $N \geqq t(M)$. If N and M/N is g.t-full then M is g.t-full.*

Proof. Let N' be an essential submodule of M. Since N is closed in M, $(N'+N)/N$ is essential in M/N. Hence $M/(N'+N)$ is torsion since M/N is $g.t$-full. Moreover, $N' \cap N$ is an essential submodule of N and N is $g.t$-full. So, $N/(N' \cap N)$ is torsion. In the exact sequence $0 \to (N'+N)/N' \to M/N' \to M/(N'+N) \to 0$, since $(N'+N)/N'$ and $M/(N'+N)$ are torsion, it follows that M/N' is torsion. Furthermore, we have $t(N' \cap N)=t(M)$ from the fact that : $N' \cap N$ is essential in N and $N \geqq t(M)$. Hence it follows that $t(N')=t(M)$ and thus M is $g.t$-full.

Proposition 5. *Let M be an R-module. The following conditions are equivalent* :

(1) *$E(M)$ is g.t-full.*

(2) *$t(E(N))=t(N)$ and $E(N)/N$ is torsion for all submodule N of $E(M)$.*

Proof. (1)⇒(2). For any submodule N of $E(M)$, $E(N)$ is $g.t$-full by Proposition 2. Hence, $t(E(N))=t(N)$ and $E(N)/N$ is torsion. (2)⇒(1). Let N be an essential submodule of $E(M)$. Then we have $E(M)=E(N)$ and hence $E(M)$ is $g.t$-full by assumption.

Proposition 6. *If M is a torsion R-module, the following are equivalent* :

(1) *M is g.t-full.*

(2) *M is semisimple.*

Proof. (1)⇒(2). Let N be a submodule of M. Then there is a submodule X of M such that $N \oplus X \leqq_e M$. Since $t(M)=M$, we see that $M=t(M)=t(N \oplus X)=t(N) \oplus t(X)=N \oplus X$. Hence $N \leqq \oplus M$. Therefore M is semisimple. (2)⇒(1) is clear.

Corollary 7. *An R-module M is g.1-full if and only if M is semisimple.*

Proof. Clear.

3. Generalized t-fullness of $_RR$

Let t be a left exact radical of R-mod. and $\mathcal{L}(t)$ the associated left Gabriel topology. Further put

$$\mathcal{L}=\{I \leqq {}_RR \mid t(R) \leqq I\}$$

This is a left linear topology. Using there concept we shall give in this section a necessary and sufficient condition for $_RR$ to be $g.t$-full

Proposition 8. *The intersection of $\mathcal{L}(t)$ and $\mathcal{L}$ is contained in $\boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is the set of all essential left ideals of R.*

Proof. Noting that $I \geqq t(R)$ if and only if $t(I)=t(R)$, by Lemma 1, we can easily prove the proposition.

Theorem 9. *The following conditions are equivarent :*

(1) *R is $g.t$-full.*

(2) $\mathcal{L}(t) \cap \mathcal{L}=\boldsymbol{\varepsilon}$.

Proof. This is clear from the proposition 8 and the definition of $g.t$-fullness.

Corollary 10. *Following conditions are equivarent :*

(1) *R is $g.1$-full.*

(2) $\mathcal{L}(1) \cap \mathcal{L}=\boldsymbol{\varepsilon}$.

(3) *R is semisimple.*

Proof. Clear.

Corollary 11. *Let (T_1, T_2, T_3) be a hereditary 3-fold torsion theory and t_1 the left exact radical with respect to (T_1, T_2). Then, R is $g.t_1$-full if and only if R is semisimple.*

Proof. It is clear from $\mathcal{L}_1 \cap \mathcal{L}_2=\mathcal{L}_1 \cap \{I \leqq R \mid I \geqq t_1(R)\}=\{R\}$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the Gabriel topologies corresponding to (T_1, T_2) and (T_2, T_3), respectively.

References

[1] J. Golan : Torsion Theories, Pitman Monographs and Surveys in Pure and Applied Mathematics 29.

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[3] B. Stenström : Rings of Quotients, Springer Verlag, Berlin Heidelberg New York, 1975.

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