COMPACT HOLOMORPHIC MAPPINGS ON LOCALLY CONVEX SPACES

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Abstract

A holomorphic mapping f on an open subset U of a locally convex space E into a locally convex space F is generally not locally bounded, much less compact. Moreover, if E and F are Banach spaces, then f is locally bounded, but it is not always compact. However, we shall show that, if E is a Schwartz space and F is a Banach space, then any holomorphic mapping on E into F is compact on E. We next describe the relation between the compactness of holomorphic mappings and that of n-homogeneous polynomials obtained from the holomorphic mappings for all n. Finally, we shall get the result that, if a holomorphic mapping on a connected open subset U of a normed space with values in a Banach space is compact at some point of U, then it is compact on U.

Introduction

When E and F are Banach spaces, R. M. Aron and M. Schottenloher [1] introduced a conception of compact holomorphic mappings on E into F. They showed that any holomorphic mapping on E into F which is compact at some point is in fact compact on E. We refer any notation used in this paper to S. Dineen [2] .

Compact holomorphic mappings on locally convex spaces

Let E and F be locally convex topological vector spaces over the complex numbers, and U be an open subset of E. A mapping from U into F is said to be holomorphic if it is continuous and Gâteaux -holomorphic. We let $H(U;F)$ denote the vector space of all holomorphic mappings from U into F. We let $P(^nE;F)$ denote the set of all continuous n-homogeneous polynomials from E into F for every positive integer n. If a mapping f from U into F is holomorphic, then for every ξ in U there exists a unique continuous n-homogeneous polynomial $\hat{d}^nf(\xi)$ from E into the completion $\tilde{F}$ of F for every nonnegative integer n such that

$$f(\xi+y)=\sum_{n=0}^{\infty}\frac{\hat{d}^nf(\xi)}{n!}(y)$$

for all y in some neighborhood of zero in E. Moreover, we have

$$\frac{\hat{d}^nf(\xi)}{n!}(y)=\frac{1}{2\pi i}\int_{|\lambda|=\rho_y}\frac{f(\xi+\lambda y)}{\lambda^{n+1}}d\lambda,$$

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where a positive number ρ_y is chosen so that $\xi + \{\lambda y;\ |\lambda| \leqq \rho_y\} \subset U$.

Definition 1. *Let E and F be locally convex spaces, U be an open subset of E, and $x \in U$. A mapping f from U into F is said to be compact at x if there is a neighborhood V_x of x such that $f(V_x)$ is a relatively compact subset of F. f is said to be compact on U if f is compact at all points x in U. We let $H_K(U;F)$ denote the vector space of all compact holomorphic mappings from U into F.*

If E is an infinite dimensional Banach space, the identity mapping of E is a locally bounded holomorphic mapping which is not compact. Thus, holomorphic mappings are not always compact. However, if E or F is finite dimensional, we have $H_K(E;F) = H(E;F)$. Moreover we have the following result.

Theorem 2. *If E is a Schwartz space and F is a Banach space, then $H_K(E;F) = H(E;F)$.*

Proof. Let a symbol $\|\cdot\|$ be a continuous norm on F. Let f be in $H(E;F)$. We first show that f is compact at zero. There exists a convex balanced neighborhood U of zero in E such that

(1) $\sup_{x \in U} \| f(x) \| < \infty.$

We put $M = \sup_{x \in U} \| f(x) \|$. By the Cauchy inequalities, we have

$$\left\| \frac{\hat{d}^n f(0)}{n!}(x) \right\| \leqq M$$

for every x in U and every non-negative integer n. Let L_n be the continuous symmetric n-linear mapping from E into F associated with $\frac{\hat{d}^n f(0)}{n!}$ for every positive integer n. By the polarization formula,

$$\| L_n (x_1, x_2, \cdots\cdots, x_n) \| \leqq \frac{n^n}{n!} M$$

for any $x_1, x_2, \cdots\cdots, x_n$ in U. Let p be the gauge of U. Since $\lambda y \in U$ for every complex number λ and every $y \in p^{-1}(0)$, we have

$$\| L_n (\lambda y, x_2, \cdots\cdots, x_n) \| \leqq \frac{n^n}{n!} M$$

for every $\lambda \in \mathbf{C}$ and any $x_2, \cdots\cdots, x_n \in U$. Hence, for every nonzero complex number λ and any $x_2, x_3, \cdots\cdots, x_n \in U$,

$$\| L_n (y, x_2, x_3, \cdots\cdots, x_n) \| \leqq \frac{1}{|\lambda|} \frac{n^n}{n!} M.$$

Since $\frac{1}{|\lambda|} \frac{n^n}{n!} \cdot M \to 0$ as $\lambda \to \infty$, we have

$$L_n (y, x_2, \cdots\cdots, x_n) = 0$$

for every $y \in p^{-1}(0)$ and any $x_2, \cdots\cdots, x_n \in E$. Since L_n is symmetric, hence, we have

$$L_n (x_1, x_2, \cdots\cdots, x_n) = 0$$

for any $x_1, x_2, \cdots\cdots, x_n \in E$ such that $p(x_i) = 0$ for some i. Let $x \in E$ and $y \in p^{-1}(0)$. Then we have

$$\frac{\hat{d}^n f(0)}{n!}(x+y) = L_n(\underbrace{x+y, x+y, \cdots\cdots, x+y}_{n \text{ times}})$$

$$= \sum_{k=0}^{n} \binom{n}{k} L_n (\underbrace{x, \cdots\cdots, x}_{n-k \text{ times}}, \underbrace{y, \cdots\cdots, y}_{k \text{ times}})$$

$$=L_n(\underbrace{x, x, \cdots\cdots, x}_{n \text{ times}})=\frac{\hat{d}^n f(0)}{n!}(x).$$

Hence we have

$$(2) \quad f(x+y)=\sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!}(x+y)=\sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!}(x)=f(x)$$

for every $x\in E$ and every $y\in p^{-1}(0)$.

Let E_p be the quotient space $E/p^{-1}(0)$ with the norm induced from the seminorm p, and Q_p be the quotient map $E\to E_p$. By (1) and (2), there is a holomorphic mapping $\tilde{f}$ from E_p into F such that $f=\tilde{f}\circ Q_p$ on E. Since E is a Schwartz space, we can choose a convex balanced neighborhood V of zero in E such that $Q_p(V)$ is a precompact subset of E_p. We may suppose $V\subset\frac{1}{2}U$. Since $\tilde{f}$ is holomorphic and bounded on $\frac{1}{2}U$, it is uniformly continuous on $\frac{1}{2}U$. Hence $\tilde{f}(Q_p(V))$ is precompact, and since F is complete the subset is relatively compact. Thus $f(V)$ is a relatively compact subset of F, since $f(V)=\tilde{f}(Q_p(V))$. Hence f is compact at zero. By using any translation map of E, we can show that f is compact at every x in U. Consequently, $f\in H_K(E;F)$ and this completes the proof.

Let E and F be locally convex spaces. Let u be a continuous linear map from E into F. We note that u is holomorphic. If $H_K(E;F)=H(E;F)$, then u is compact. If $H_K(E;F)=H(E;F)$ for any Banach space F, then all continuous linear maps from E into F are compact for any Banach space F, and it follows from the known results in the theory of Schwartz spaces that E is a Schwartz space.

Let E, F be locally convex spaces and U be an open subset of E. Now we shall describe the relation between the compactness of holomorphic mapping f on U into F and that of $\hat{d}^n f(\xi)$ for $\xi\in U$ and any positive integer n.

Proposition 3. *Let E be a locally convex space, F be a complete locally convex space, and U be an open subset of E. Let $\xi\in U$ and $f\in H(U;F)$. If f is compact at ξ, then $\hat{d}^n f(\xi)$ is compact on E for all n.*

Proof. We may suppose that $\xi=0$. Since f is compact at zero in E, we can choose a balanced convex neighborhood V, $2V\subset U$, of zero in E such that $f(2V)$ is a relatively compact subset of F. Since f is continuous, by the Riemann integral, for every $x\in V$ we have

$$\frac{\hat{d}^n f(0)}{n!}(x)=\frac{1}{2\pi i}\int_{|\lambda|=1}\frac{f(\lambda x)}{\lambda^{n+1}}d\lambda \qquad (\text{we put } \lambda=e^{i\theta})$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta}x)}{e^{in\theta}}\,d\theta$$

$$=\lim_{k\to\infty}\frac{1}{2\pi}\sum_{m=1}^{k}\frac{f(e^{2\pi im/k}x)}{e^{2\pi inm/k}}\frac{2\pi}{k}$$

$$=\lim_{k\to\infty}\frac{1}{k}\sum_{m=1}^{k}\frac{f(e^{2\pi im/k}x)}{e^{2\pi inm/k}}$$

for all n. Since F is complete and $f(V)$ is relatively compact, the closed convex balanced hull A of $f(V)$ is compact. For every positive integer k,

$$\frac{f(e^{2\pi im/k}x)}{e^{2\pi inm/k}}$$

is included in A for $m=1, 2, \cdots\cdots, k,$ and hence

$$\sum_{m=1}^{k} \frac{f(e^{2\pi im/k}x)}{e^{2\pi inm/k}}$$

is included in kA. It follows that

$$\frac{1}{k}\sum_{m=1}^{k} \frac{f(e^{2\pi im/k}x)}{e^{2\pi inm/k}}$$

is included in A for every positive integer k. Since A is closed, we have $\frac{\hat{d}^n f(0)}{n!}(x) \in A$ for every $x \in V$. Hence $\frac{\hat{d}^n f(0)}{n!}(V) \subset A$ for all n. Since A is compact, $\frac{\hat{d}^n f(0)}{n!}(V)$ is relatively compact and hence $\hat{d}^n f(0)(V)$ is relatively compact for all n. Consequently $\hat{d}^n f(0) \in H_K(E;F)$ for all n. This completes the proof.

Moreover, the above proof shows that $\hat{d}^n f(0)(V)$ is contained in the vector space spanned by $f(V)$ for all n.

LEMMA 4. *Let E be a locally convex space, F be a complete locally convex space, and U be an open subset of E. Let $f \in H(U;F)$, $\xi \in U$, and V be a balanced convex neighborhood of zero in E such that $\xi + V$ is contained in U and $f(\xi + V)$ is bounded. Let $\{a_m\}_{m=1}^{\infty}$ be a sequence in $\xi + \rho V$ for some ρ, $0<\rho<1$. If the sequence $\left\{\frac{\hat{d}^n f(\xi)}{n!}(a_m)\right\}_{m=1}^{\infty}$ is convergent for all n, then the sequence $\{f(\xi + a_m)\}_{m=1}^{\infty}$ is also convergent in F.*

PROOF. For every non-negative integer n, let $y_n \in F$ be the limit point of a sequence $\left\{\frac{\hat{d}^n f(\xi)}{n!}(a_m)\right\}_{m=1}^{\infty}$. We first suppose that F is a Banach space. Let a symbol $\|\cdot\|$ be a continuous norm on F. Since $f(\xi + V)$ is bounded,

$$\sup_{x \in V} \|f(\xi + x)\| = M < \infty.$$

By the Cauchy inequalities, we have

$$\sup_{x \in \rho V} \left\| \frac{\hat{d}^n f(\xi)}{n!}(x) \right\| \leqq \rho^n M$$

for all n. Hence for arbitrary positive number ε, there exists an integer n_0 such that

$$(3) \quad \sum_{n=r}^{\infty} \sup_{x \in \rho V} \left\| \frac{\hat{d}^n f(\xi)}{n!}(x) \right\| < \varepsilon$$

for every integer $r, r > n_0$. Since

$$\left\| \frac{\hat{d}^n f(\xi)}{n!}(a_m) \right\| \leqq \sup_{x \in \rho V} \left\| \frac{\hat{d}^n f(\xi)}{n!}(x) \right\| \leqq \rho^n M$$

for every positive integer m, we have

$$\|y_n\| \leqq \rho^n M$$

for all n. Hence we have

$$\left\| \sum_{n=0}^{\infty} y_n \right\| \leqq \sum_{n=0}^{\infty} \|y_n\| \leqq \sum_{n=0}^{\infty} \rho^n M < \infty.$$

Thus the series $\sum_{n=0}^{\infty} y_n$ converges absolutely to some point y_0 in F. There is an integer n_1 such that

$$(4) \quad \left\| \sum_{n=r}^{\infty} y_n \right\| \leqq \sum_{n=r}^{\infty} \|y_n\| < \varepsilon$$

for every r, $r>n_1$. We put $n_2=\max(n_0, n_1)$. Since, for every n, $\dfrac{\hat{d}^n f(\xi)}{n!}(a_m)\to y_n$ as $m\to\infty$, there is an integer m_0 such that

$$(5)\quad \sum_{n=0}^{n_2}\left\| \frac{\hat{d}^n f(\xi)}{n!}(a_m)-y_n \right\| <\varepsilon$$

for every m, $m>m_0$. For every integer $m>m_0$, by (3), (4) and (5) we have

$$\| f(\xi+a_m)-y_0 \| = \left\| \sum_{n=0}^{\infty} \frac{\hat{d}^n f(\xi)}{n!}(a_m)-\sum_{n=0}^{\infty} y_n \right\|$$

$$= \left\| \sum_{n=0}^{n_2}\left\{ \frac{\hat{d}^n f(\xi)}{n!}(a_m)-y_n \right\} + \sum_{n=n_2+1}^{\infty} \frac{\hat{d}^n f(\xi)}{n!}(a_m)-\sum_{n=n_2+1}^{\infty} y_n \right\|$$

$$\leqq \sum_{n=0}^{n_2}\left\| \frac{\hat{d}^n f(\xi)}{n!}(a_m)-y_n \right\| + \sum_{n=n_2+1}^{\infty}\left\| \frac{\hat{d}^n f(\xi)}{n!}(a_m) \right\| + \sum_{n=n_2+1}^{\infty}\| y_n \|$$

$$<\varepsilon+\varepsilon+\varepsilon=3\,\varepsilon.$$

Hence $f(\xi+a_m)\to y_0$ as $m\to\infty$. Thus, if F is a Banach space, the sequence $\{f(\xi+a_m)\}_{m=1}^{\infty}$ is convergent in F.

Next, let F be a complete locally convex space. Let p be an arbitrary continuous seminorm on F. Let F_p be the quotient space $F/p^{-1}(0)$ with the norm induced from the seminorm p, and Q_p be the quotient map $F\to F_p$. We let $\tilde{F}_p$ denote the completion of the normed space F_p. Then $Q_p\circ f\in H(U;F_p)$, and

$$(\hat{d}^n(Q_p\circ f)(\xi))(x)=Q_p(\hat{d}^n f(\xi)(x))$$

for nonnegative integer n and all x in E. Hence, for all n, the sequence $\left\{ \dfrac{\hat{d}^n(Q_p\circ f)(\xi)}{n!}(a_m) \right\}_{m=1}^{\infty}$ is convergent in $\tilde{F}_p$, and $Q_p\circ f$ satisfies the conditions of this lemma. Hence, by the above, the sequence $\{Q_p\circ f(\xi+a_m)\}_{m=1}^{\infty}$ is convergent in $\tilde{F}_p$. Hence, it is a Cauchy sequence of F_p. Since p is an arbitrary continuous seminorm, it follows that $\{f(\xi+a_m)\}_{m=1}^{\infty}$ is a Cauchy spquence of F. Since F is complete, $\{f(\xi+a_m)\}_{m=1}^{\infty}$ is convergent in F. This completes the proof.

Proposition 5. *Let E be a locally convex space, F be a Fréchet space, and U be an open subset of E. Let $f\in H(U;F)$, $\xi\in U$ and V be a balanced convex neighborhood of zero in E such that $\xi+V$ is contained in U and $f(\xi+V)$ is bounded. Let B be a subset of ρV for some ρ, $0<\rho<1$. If $\hat{d}^n f(\xi)(B)$ is a relatively compact subset of F for all n, then $f(\xi+B)$ is also relatively compact.*

Proof. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence of B. Since F is a Fréchet space and $\hat{d}^1 f(\xi)(B)$ is relatively compact, we can find a subsequence $\{x_{1,m}\}_{m=1}^{\infty}$ of $\{x_m\}_{m=1}^{\infty}$ such that $\{\hat{d}^1 f(\xi)(x_{1,m})\}_{m=1}^{\infty}$ is convergent. By induction with respect to n, we can find a subsequence $\{x_{n,m}\}_{m=1}^{\infty}$ of $\{x_m\}_{m=1}^{\infty}$ for every positive integer n such that $\{x_{n,m}\}_{m=1}^{\infty}$ is a subsequence of $\{x_{n-1,m}\}_{m=1}^{\infty}$ and $\{\hat{d}^n f(\xi)(x_{n,m})\}_{m=1}^{\infty}$ is convergent for all n. Then $\{x_{m,m}\}_{m=1}^{\infty}$ is a subsequence of $\{x_m\}_{m=1}^{\infty}$ and by the construction of $\{x_{m,m}\}_{m=1}^{\infty}$, $\{\hat{d}^n f(\xi)(x_{m,m})\}_{m=1}^{\infty}$ is convergent for all n. By lemma 4, $\{f(\xi+x_{m,m})\}_{m=1}^{\infty}$ is convergent in F. Thus the sequences of $f(\xi+B)$ contain convergent subsequences. This implies that $f(\xi+B)$ is a relatively compact subset of F. This completes the proof.

Proposition 6. *Let E, F, U, f, ξ and V satify the same conditions of proposition 5. Let $\hat{d}^n f(\xi)$ be compact on E for all n. If B is bounded and a subset of ρV for some ρ with $0<\rho<1$, then $f(\xi+B)$ is a relatively compact subset of F.*

Proof. For every positive integer n, we can choose a balanced convex neighborhood W of zero in E such that

$\hat{d}^n f(\xi)(W)$ is relatively compact. Since B is bounded, there exists a positive number λ such that $\lambda B \subset W$. Since $\hat{d}^n f(\xi)(\lambda B) \subset \hat{d}^n f(\xi)(W)$, $\hat{d}^n f(\xi)(\lambda B)$ is relatively compact. Since $\hat{d}^n f(\xi)$ is an n-homogeneous polynomial, we have $\hat{d}^n f(\xi)(\lambda B) = \lambda^n \hat{d}^n f(\xi)(B)$, and hence

$$\hat{d}^n f(\xi)(B) \subset \frac{1}{\lambda^n} \hat{d}^n f(\xi)(W).$$

Since $\hat{d}^n f(\xi)(W)$ is relatively compact, $\frac{1}{\lambda^n}\hat{d}^n f(\xi)(W)$ is also relatively compact. Hence $\hat{d}^n f(\xi)(B)$ is relatively compact for all n. Hence, by proposition 5, $f(\xi + B)$ is relatively compact. This completes the proof.

PROPOSITION 7. *Let E be a normed space, F be a Banach space, and U be an open subset of E. Let* $f \in H(U;F)$ *and* $\xi \in U$. *If* $\hat{d}^n f(\xi)$ *is compact on E for all n, then f is compact at* ξ.

PROOF. Let B be the closed unit ball of E. Since f is continuous, there is a positive number λ such that $f(\xi + \lambda B)$ is bounded. Since $\rho\lambda B$ is bounded and a subset of λB for every ρ with $0 < \rho < 1$, by proposition 6, $f(\xi + \rho\lambda B)$ is relatively compact. Since $\xi + \rho\lambda B$ is a neighborhood of ξ, f is compact at ξ. This completes the proof.

Proposition 3 and proposition 7 imply the following result.

PROPOSITION 8. *Let E be a normed space, F be a Banach space, and U be an open subset of E. Let* $f \in H(U;F)$ *and* $\xi \in U$. *f is compact at* ξ *if and only if* $\hat{d}^n f(\xi)$ *is compact on E for all n.*

PROPOSITION 9. *Let E be a normed space, F be a Banach space and U be a connected open subset of E. Let* $f \in H(U;F)$. *If f is compact at some point of U, then f is compact on U.*

PROOF. We put $A = \{x \in U; f \text{ is compact at } x\}$. By the hypothesis of this proposition and definition 1, A is a nonempty open subset of U. We suppose that A is not relatively closed in U. Then, there exists a point x_0 in $(\bar{A} \setminus A) \cap U$, where $\bar{A}$ is the point set closure of A. Let V be the open unit ball of E. Then, we can find a positive number λ such that $x_0 + 2\lambda V \subset U$ and $f(x_0 + 2\lambda V)$ is bounded. Since $x_0 + \lambda V$ is a neighborhood of x_0, there is a point ξ in $A \cap (x_0 + \lambda V)$. Then, we have

$$\xi + \lambda V \subset x_0 + \lambda V + \lambda V = x_0 + 2\lambda V.$$

Hence, $\xi + \lambda V \subset U$ and $f(\xi + \lambda V)$ is bounded. Since λV is an open ball of E, there exists a positive number ρ with $0 < \rho < 1$ such that $x_0 \in \xi + \rho\lambda V$. Since f is compact at ξ, $f(\xi + \lambda V)$ is bounded and $\rho\lambda V$ is a bounded subset of λV, by proposition 3 and proposition 6, $f(\xi + \rho\lambda V)$ is relatively compact. Since $\xi + \rho\lambda V$ is an open subset containing x_0, it is a neighborhood of x_0. Hence f is compact at x_0, and so $x_0 \in A$. This contradicts the assumption. Hence, A must be relatively closed in U. It follows from the connectedness of U that $A = U$. This implies that f is compact on U.

We get the following result by the above.

THEOREM 10. *Let E be a normed space, F be a Banach space and U be a connected open subset of E. If* $f \in H(U;F)$, *then the following are equivalent:*

(a) *f is compact on U,*

(b) *f is compact at some point of U,*

(c) $\hat{d}^n f(\xi)$ *is compact on E for every point* ξ *in U and all n,*

(d) *for some point* ξ *of* U, $\hat{d}^n f(\xi)$ *is compact on* E *for all* n.

References

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