On Some Prime Modules

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Abstract

In [3] Page has defined the notion of prime module. Recently, L. Bican, P. Jambor, T. Kepka and P. Němec [1] have also defined the prime module by different notion. In this paper, with reference to the latter definition, we shall define the notions of several kinds of prime modules, that is, a strongly prime module, a weakly prime module, an E-prime module, an E'-prime module and an E''-prime module. And then, we will investigate the relations between these prime modules.

§ 0. Introduction

In this paper, with reference to the notion of prime module in the sense of Bican et al., we shall define the notions of several kinds of prime modules, that is, a strongly prime module, a weakly prime module, an E-prime module, an E'-prime module and an E''-prime module.

We shall prove in Proposition 1.2 that the prime module in the sense of Bican et al. is the prime module in the sense of Page. But the converse is not true (Example 7.1). Nevertheless, we shall show in Proposition 1.3, if R is semisimple artinian, the converse holds. And also we shall prove in Proposition 1.2 that strongly prime modules are prime, prime modules are weakly prime, strongly prime modules are nothing but E'-prime and also are E-prime, and in Proposition 6.3 that, over a left noetherian ring R, every injective R-module is a direct sum of weakly prime modules (E-prime modules, E''-prime modules).

§ 1. Definitions

Throughout this note R is an associative ring with identity, a module means a unital left R-module and R-mod stands for the category of unital left R-modules. As usual, $E(M)$ will denote the injective hull of a module M.

A preradical r for R-mod is a subfunctor of the identity functor of R-mod.

For a module Q, define a radical k_Q as

$$k_Q(M) \cup = \{\mathrm{Ker}\ f \mid f \in \mathrm{Hom}\ (M,\ Q)\}$$

for each module M. As is well known, k_Q is a unique maximal one of those preradicals k for R-mod satisfying $k(Q)=0$.

For two submodules A and B of a module M, we put

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$$A*_M B = \sum \{f(A) \mid f \in \text{Hom}(M, B)\}.$$

We denote by Ann(M) the annihilator ideal of a module M.

Page called a module M *prime* if $Ann(M) = Ann(N)$ for every non-zero submodule N of M. And Bican et al. called a module M *prime* if $k_M = k_N$ for every non-zero submodule N of M.

In the following, P-prime module means the prime module in the sense of Page, and the prime module means the one in the sense of Bican et al..

We now define the notions of several kinds of prime modules. We call a module M

(1) *strongly prime(S-prime)* if *$E(M)$ is prime*. Moreover we call a module M

(2) *weakly prime(W-prime)* if $k_M = k_{E(N) \cap M}$,

(3) *E-prime* if $k_{E(M)} = k_{E(N)}$,

(4) *E′-prime* if $k_{E(M)} = k_N$,

(5) *E″-prime* if $k_M = k_{E(N)}$,

for every non-zero submodule N of M.

Example 1. 1.

(1) *A module $_ZQ$ is P-prime, where Z is the ring of integers and Q is the additive group of rational numbers.*

(2) *Each simple module is prime and also E-prime.*

(3) *The injective hull of each simple module is W-prime.*

(4) *A simple injective module is S-prime, E′-prime and E″-prime.*

Proposition 1. 2.

(1) *Every prime module is P-prime.*

(2) *Every S-prime module is prime and every prime module is W-prime.*

(3) *Every S-prime module is nothing but E′-prime module.*

(4) *Every S-prime module is E-prime.*

Proof. (1) Let M be a prime module. For any non-zero submodule N of M, we obtain $k_M(R) = k_N(R)$ from the definition. It is well known that, for a module L, $k_L(R) = \text{Ann}(L)$ holds. Thus Ann (M) = Ann (N) for any non-zero submodule N of M. Therefore M is P-prime.

(2) Let M be S-prime and N any non-zero submodule of M. Then the relation $E(M) \supseteqq M \supseteqq N$ implies $k_{E(M)} \leqq k_M \leqq k_N$. Since M is S-prime, $k_{E(M)} = k_N$. Therefore we obtain $k_M = k_N$. That is, M is prime. Now let M be prime and N any non-zero submodule of M. Then $E(N) \cap M$ is a non-zero submodule of M, and so the primeness of M implies $k_M = k_{E(N) \cap M}$. Hence M is W-prime.

(3) Suppose that M is an S-prime module. Then $E(M)$ is prime and so we have $k_{E(M)} = k_N$ for all non-zero submodule N of M, namely, M is E'-prime.

Conversely, let M be E'-prime. For any non-zero submodule N of $E(M)$, $M \cap N$ is a non-zero submodule of M and so we have $k_{E(M)} \leqq k_N \leqq k_{M\cap N}$. However, since M is E'-prime, we have $k_{E(M)} = k_{M\cap N}$. Therefore, $k_{E(M)} = k_N$ and hence M is S-prime.

(4) Let M be S-prime and N any non-zero submodule of M. Then $N \subseteqq E(N) \subseteqq E(M)$ and so we have $k_{E(M)} \leqq k_{E(N)} \leqq k_N$. Since M is S-prime, $k_{E(M)} = k_N$. This relation implies that $k_{E(M)} = k_{E(N)}$. Namely, M is E-prime.

Proposition 1. 3. *Let M be a module.*

(1) *If R is semisimple artinian, then every P-prime module is prime.*

(2) *If M is prime and injective, then it is S-prime.*

(3) *If M is W-prime and any non-zero submodule N of M is injective, then M is prime.*

(4) *If M is E-prime and any non-zero submodule N of M is injective, then M is S-prime.*

Proof. (1) It is well known that, for a preradical r and a projective module P, $r(P) = r(R)P$. Now, let M be a P-prime module and N any non-zero submodule. By Ann (M) = Ann (N), we have $k_M(R) = k_N(R)$. For any module A, we have $k_M(A) = k_M(R)A = k_N(R)A = k^N(A)$, since R is semisimple artinian. Therefore $k_M = k_N$ for any non-zero submodule N of M, which means M is prime.

(2) Since M is injective, we have $E(M) = M$. Therefore, $E(M)$ is prime and M is S-prime.

(3) It follows from assumption that $k_M = k_{E(N)\cap M} = k_{N\cap M} = k_N$ for every non-zero submodule N of M. Hence M is prime.

(4) Let N be any non-zero submodule of $E(M)$ $(= M)$. Since M is E-prime and N is injective, $k_{E(M)} = k_{E(N)} = k_N$ is obtained. Namely, M is S-prime.

Corollary 1. 4. *The following conditions are equivalent for an injective module M:*

(1) *M is prime,*

(2) *M is S-prime.*

Corollary 1. 5. *For a module M over a semisimple artinian ring R, the following conditions are equivalent:*

(1) *M is prime,*

(2) *M is P-prime,*

(3) *M is S-prime,*

(4) *M is W-prime,*

(5) *M is E-prime,*

(6) *M is E''-prime.*

§ 2. Prime modules

The following propositions are due to Bican et al. [1]. In the following we shall give similar characterizations for our prime modules.

Proposition 2. 1. ([1, Proposition 2. 3]). *The following conditions are equivalent for a module M:*

(1) $A*_M B \neq 0$ *for all non-zero submodules* $A, B \subseteqq M$,

(2) $k_N(M) = 0$ *for every non-zero submodule* $N \subseteqq M$,

(3) *If* $0 \neq N \subseteqq M$, *then M is isomorphic to a submodule of a direct product of copies of N,*

(4) *M is prime.*

Proposition 2. 2. ([1, Proposition 2. 4]).

(1) *Let N be a non-zero submodule of M. If M is prime then N is prime.*

(2) *A module M is prime if and only if* $k_M = k_C$ *for every non-zero cyclic submodule C of M.*

(3) *Every simple module is prime and every direct sum of copies of a simple module is prime.*

§ 3. Weakly prime modules

Proposition 3. 1. *The following conditions are equivalent for a module M:*

(1) $A*_M(E(B) \cap M) \neq 0$ *for all non-zero submodules* $A, B \subseteqq M$,

(2) $k_{E(N)\cap M}(M) = 0$ *for every non-zero submodule* $N \subseteqq M$,

(3) *If* $0 \neq N \subseteqq M$, *then M is isomorphic to a submodule of a direct product of copies of* $E(N) \cap M$,

(4) *M is W-prime.*

Proof. (1) implies (2). Suppose that $k_{E(N)\cap M}(M) = N' \neq 0$ for some $0 \neq N \subseteqq M$. Then $f(N') = 0$ for every $f \in \mathrm{Hom}\,(M, E(N) \cap M)$ and so $N'*_M(E(N) \cap M) = 0$, a contradiction.

(2) implies (3). See e. g. [2, p. 408].

(3) implies (4). Let $0 \neq N \subseteqq M$ be a submodule. Obviously, $k_M \leqq k_{E(N)\cap M}$ and $k_{E(N)\cap M}(E(N) \cap M) = 0$. Since M is isomorphic to a submodule of a direct product of copies of $E(N)\cap M$, we have $k_{E(N)\cap M}(M) = 0$.
Consequently, $k_M \geqq k_{E(N)\cap M}$. Therefore $k_M = k_{E(N)\cap M}$ and M is W-prime.

(4) implies (1). Suppose that $A*_M(E(B) \cap M) = 0$ for some $A(\neq 0)$, $B(\neq 0) \subseteqq M$. Then $f(A) = 0$ for all $f \in \mathrm{Hom}\,(M, E(B) \cap M)$ and so $0 \neq A \subseteqq k_{E(B)\cap M}(M) = k_M(M) = 0$, a contradiction.

Proposition 3. 2. *A module M is W-prime if and only if $k_M = k_{E(C)\cap M}$ for every non-zero cyclic submodule C of M.*

Proof. The "only if" part is clear. To show the "if" part, let N be any non-zero submodule of M and x a non-zero element in N. Then $M \supseteqq N \supseteqq Rx$ and so $M \supseteqq E(N) \cap M \supseteqq E(Rx) \cap M$. This implies $k_M \leqq k_{E(N)\cap M} \leqq k_{E(Rx)\cap M}$. By assumption $k_M = k_{E(Rx)\cap M}$ and hence $k_M = k_{E(N)\cap M}$. Thus M is W-prime.

Proposition 3. 3. *Let M be a W-prime module and N a non-zero injective submodule of M. Then N is W-prime.*

Proof. For every non-zero submodule L of N, we have $k_M = k_{E(L)\cap M}$. Since $M \supseteqq N \supseteqq E(L)$, $k_{N\cap E(L)} = k_{E(L)} = k_{E(L)\cap M} = k_M = k_{E(N)\cap M} = k_{N\cap M} = k_N$ and so N is W-prime.

Proposition 3. 4. *Let M be a module. Then $E(M)$ is W-prime if and only if $k_{E(M)} = k_{E(N)}$ for every non-zero submodule N of $E(M)$.*

Proof. This follows from the relations $E(N) \subseteqq E(E(M)) = E(M)$ and $E(N) \cap E(M) = E(N)$.

Corollary 3. 5. *The following conditions are equivalent for an injective module M:*

(1) *M is W-prime,*

(2) *M is E-prime,*

(3) *M is E''-prime.*

Proposition 3. 6. *Let S be a simple module. Then*

(1) *S is W-prime,*

(2) *$\sum\oplus S$ is W-prime,*

(3) *$E(S)$ is W-prime.*

Proof. The assertions of (1) and (2) follow from [1, Proposition 2. 4] and Proposition 1. 2 (2).

(3) For every non-zero submodule N of $E(S)$, we have $E(N) = E(S)$. Hence $k_{E(N)} = k_{E(S)}$ and so $E(S)$ is W-prime by Proposition 3. 4.

§ 4. Strongly prime modules

Proposition 4. 1. *The following conditions are equivalent for a module M:*

(1) *$A*_{E(M)}B \neq 0$ for all non-zero submodules A, $B \subseteqq E(M)$,*

(2) *$k_N(E(M)) = 0$ for every non-zero submodule $N \subseteqq E(M)$,*

(3) *If* $0 \neq N \subseteqq E(M)$, *then* $E(M)$ *is isomorphic to a submodule of a direct product of copies of* N,

(4) M *is S-prime.*

Proof. This follows from Proposition 2. 1.

Proposition 4. 2. *The following conditions are equivalent for a module* M*:*

(1) $A*_{E(M)} B \neq 0$ *for all non-zero submodules* $A, B \subseteqq M$,

(2) $k_N(E(M)) = 0$ *for every non-zero submodule* $N \subseteqq M$,

(3) *If* $0 \neq N \subseteqq M$, *then* $E(M)$ *is isomorphic to a submodule of a direct product of copies of* N,

(4) M *is S-prime.*

Proof. This is similar to the proof of Proposition 3. 1.

Proposition 4. 3. *Let* N *be a non-zero submodule of an S-prime module* M. *Then* N *is S-prime.*

Proof. This follows from Proposition 2. 2 (1).

Proposition 4. 4. *Let* N *be an essential submodule of a module* M. *Then,* N *is S-prime if and only if* M *is S-prime.*

Proof. This is clear from the fact that $E(N) = E(M)$.

Proposition 4. 5. *A module* M *is S-prime if and only if* $k_{E(M)} = k_C$ *for every non-zero cyclic submodule* C *of* M.

Proof. The proof of this proposition is similar to that of Proposition 3. 2. This also follows from Proposition 2. 2 (2).

§ 5. E-prime modules

Proposition 5. 1. *The following conditions are equivalent for a module* M*:*

(1) $A*_{E(M)}E(B) \neq 0$ *for all non-zero submodules* $A, B \subseteqq M$,

(2) $k_{E(N)}(E(M)) = 0$ *for every non-zero submodule* $N \subseteqq M$,

(3) *If* $0 \neq N \subseteqq M$, *then* $E(M)$ *is isomorphic to a submodule of a direct product of copies of* $E(N)$,

(4) M *is E-prime.*

Proof. The proof of this proposition is similar to that of Proposition 3. 1.

Proposition 5. 2. *A module M is E-prime if and only if* $k_{E(M)} = k_{E(}$ *for every non-zero cyclic submodule C of M.*

Proof. The proof of this proposition is similar to that of Proposition 3. 2.

Proposition 5. 3. *If M is E-prime and N is a non-zero submodule, then N is E-prime.*

Proof. This is clear.

Proposition 5. 4. *Let N be an essential submodule of M. Then N is E-prime if and only if M is E-prime.*

Proof. Suppose that N is E-prime. Then for any non-zero submodule K of M, we have $E(N \cap K) \subseteqq E(K) \subseteqq E(M) = E(N)$ and hence $k_{E(N \cap K)} \geqq k_{E(K)} \geqq k_{E(M)} = k_{E(N)}$. From the assumption it follows that $k_{E(N \cap K)} = k_{E(N)}$ and hence $k_{E(K)} = k_{E(M)}$. Thus M is E-prime. The "if" part follows from Proposition 5. 3.

Corollary 5. 5. *A module M is E-prime if and only if E(M) is E-prime.*

§ 6. Some supplements

Proposition 6. 1. *If M is injective and uniform, then M is W-prime.*

Proof. For any non-zero submodule N of M, it is clear that $E(M) = M = E(N)$. Hence we have $k_{M \cap E(N)} = k_M$.

Proposition 6. 2. *Every simple injective module is S-prime.*

Proof. Let S be a simple injective module and N a non-zero submodule of $S = E(S)$. Then $S = N$ and so we have $E(S) = N$. Therefore $k_{E(S)} = k_N$.

Proposition 6. 3. *Let R be a left noetherian ring. Then every injective R-module is a direct sum of W-prime modules (E-prime modules, E″-prime modules).*

Proof. Every injective module M can be represented as a direct sum of indecomposable injective modules M_α. In this case, each M_α is injective and uniform. Hence it is W-prime (E-prime, E''-prime).

§ 7. Examples

Example 7. 1. *Let Z be the ring of integers and Q the additive group of rational numbers. Then* ${}_ZQ$ *is P-prime and W-prime but not prime.*

Proof. It is clear that ${}_ZQ$ is P-prime. And it is well known that ${}_ZQ$ is injective and uniform. Hence $_ZQ$ is W-prime. However, since Hom $(_ZQ, {}_ZZ) = 0$, we have $k_Z(Q) \neq 0$. Therefore $_ZQ$ is not prime.

Example 7. 2. *Let S be a simple module with $E(S) \neq S$. Then S is prime and E-prime, but not S-prime.*

Proof. Assume that S is S-prime. There is a non-zero homomorphism f of $E(S)$ to S such that $f(S) \neq 0$. We can claim that this homomorphism is an isomorphism. Hence $E(S) = S$, a contradiction. Now, we shall claim that S is E-prime. Let S' be a non-zero submodule of S. Then $S = S'$ and $k_{E(S)} = k_{E(S')}$. This implies that S is E-prime.

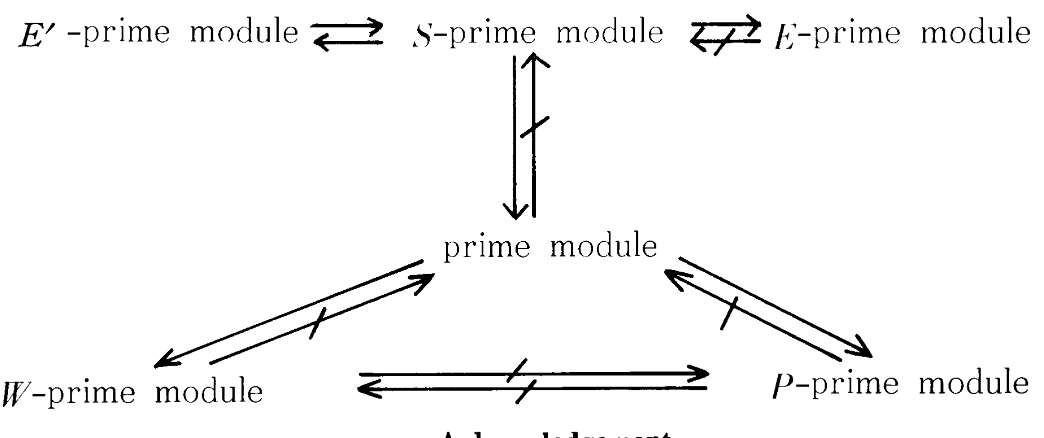
Example 7. 3. (H. Katayama). *Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \middle| \, a, b \in K \right\}$, where K is a field. Then $_RR$ is W-prime but not P-prime.*

Proof. It is well known that $_RR$ is injective and uniform module. Therefore $_RR$ is W-prime. Let $J = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \middle| \, b \in K \right\}$. Then Ann$(R) = 0$ and Ann$(J) = J$. Hence $_RR$ is not P-prime.

Example 7. 4. (H. Katayama). *Let Z be the ring of integers and $M = Z \times Q$, where Q the additive group of rational numbers. Then M is P-prime but not W-prime.*

Proof. It is clear that $_ZM$ is P-prime. Consider the cyclic submodule $C = Z \times 0$ of M. As is well known $E(C) = Q \times 0$, and so $E(C) \cap M = Z \times 0$. Put $x = (0, 1) \in M$. It is easy to show that $f(x) = 0$ for all $f \in$ Hom $(M, E(C) \cap M)$. Hence M is not W-prime.

After all we obtain the following diagram.



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