ON BOUNDING SUBSETS OF LOCALLY CONVEX SPACES

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Abstract

In the present paper, we shall describe an example of a non-separable and non-reflexive Banach space whose bounding subsets are compact subsets. Moreover, We shall prove that, if C_E and C_F are respectively bounding subsets of metrizable locally convex spaces E and F, then the subset $C_E \times C_F$ of Cartesian product $E\times F$ is also a bounding subset of $E\times F$.

Introduction

Let E be a locally convex topological vector space over the field C of complex numbers. If E is finite dimensional, every bounding subset of E is compact. Moreover, S. Dineen [3] showed that the bounding subsets of a separable or reflexive Banach space are the compact subsets. On the other hand, S. Dineen [4] proved that there is a non-compact bounding subset of l^∞.

Bounding subsets of locally convex spaces

Let E be a complex locally convex space, and U be an open set in E. When F is a complex locally convex space, $H(U;F)$ denotes the set of all holomorphic mappings on U into F. If $F=\mathbf{C}$, $H(U;\mathbf{C})$ is briefly denoted by $H(U)$.

DEFINITION 1. *A closed subset C of U is said to be a bounding subset of U if*

$$\|f\|_C = \sup_{x\in C} |f(x)| < +\infty$$

for each $f \in H(U)$.

PROPOSITION 2. *Let F be a locally convex space. A closed subset C of U is a bounding subset of U if and only if f(C) is a bounded subset of F for each $f \in H(U;F)$.*

PROOF. First, we suppose that a closed subset C of U is bounding. Now we assume that there is a holomorphic mapping $f \in H(U;F)$ such that $f(C)$ is unbounded in F. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ in C such that $\{f(x_n)\}_{n=1}^{\infty}$ is unbounded. Hence, we have a continuous linear mapping φ on F such that

$$\sup_n |\varphi(f(x_n))| = \infty .$$

Since $\varphi\circ f \in H(U)$, this contradicts the hypothesis that C is a bounding subset of U.

Conversely, we suppose that $f(C)$ is a bounded subset of F for each $f \in H(U;F)$. we assume that C is not a bounding subset of U. Then there is a holomorphic function $f \in H(U)$ such that

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$\|f\|_C=\infty$. Let p be a continuous seminorm on F. we define a holomophic mapping $h \in H(U;F)$ by $h(x) = f(x)\cdot a$ for $x \in U$, where a is a point of E such that $p(a) \neq 0$. Then we have

$$\sup_{x\in C} p\circ h(x) = \|f\|_C \cdot p(a) = \infty.$$

This contradicts the fact that $h(C)$ is bounded in F. This completes the proof.

S. Dineen [3] showed that every bounding subset of a separable or reflexive Banach space is a compact subset. However, we have a non-separable and non-refexive Banach space whose bounding subsets are compact subsets. Now we shall describe it.

Let I be an uncountable index set. $l^\infty(I)$ denotes the set of all complex valued bounded functions on I. We endow $l^\infty(I)$ with the supremum norm $\|\cdot\|$. Then, $l^\infty(I)$ is a Banach space. Let $x \in l^\infty(I)$. When x_i denotes $x(i)$ for $i \in I$, we can represent x by $(x_i)_{i\in I}$. Let $C_0(I)$ be the closure of the subspace

$$\{(x_i)_{i\in I} \in l^\infty(I) \text{ ; there exists a finite subset } J \text{ of } I \text{ such that } x_i = 0 \text{ for every } i \in I\text{-}J\}$$

Then $C_0(I)$ is a closed subspace of $l^\infty(I)$. Hence, $C_0(I)$ is a Banach space, equipped with the norm $\|\cdot\|$ induced from $l^\infty(I)$.

LEMMA 3. *Let N be the set of natural numbers. Let $C_0(N)$ be the linear space*

$$\{(x_n)_{n=1}^\infty \text{ ; } x_n \in \mathbf{C},\ n=1,2,\ldots,\ \lim_{n\to\infty} x_n = 0\}$$

with the supremum norm $\|\cdot\|$. Let ε_n be a nonnegative number for $n=1, 2, \ldots$ such that $\varepsilon_n \to 0$ as $n\to\infty$. Then the subset

$$V = \{(x_n)_{n=1}^\infty \in C_0(N) \text{ ; } |x_n| \leqq \varepsilon_n \text{ for } n=1,2,\ldots\}$$

is compact.

PROOF. Let $(x_n^k)_{n=1}^\infty = x^k \in V$ for $k=1, 2 \ldots$. Since the sequence $\{x_1^k\}_{k=1}^\infty$ is bounded in $\mathbf{C}$, we can select a convergent subsequence $\{x_1^{1k}\}_{k=1}^\infty$ of the sequence $\{x_1^k\}_{k=1}^\infty$. Next, we can select a convergent subsequence $\{x_2^{2k}\}_{k=1}^\infty$ of the sequence $\{x_2^{1k}\}_{k=1}^\infty$. Similarly, we can inductively select a convergent subsequence $\{x_{m+1}^{(m+1)k}\}_{k=1}^\infty$ of the sequence $\{x_{m+1}^{mk}\}_{k=1}^\infty$ for $m=1, 2, \ldots$. Thus we can take a subsequence $\{x^{mk}\}_{k=1}^\infty$ $(m=1, 2, \ldots)$ of the sequence $\{x^k\}_{k=1}^\infty$, with the following properties

(i) the sequence $\{x_m^{mk}\}_{k=1}^\infty$ converges, for $n=1, 2, \ldots$,

(ii) the sequence $\{x^{mk}\}_{k=1}^\infty$ is a subsequence of the sequence $\{x^{(m-1)k}\}_{k=1}^\infty$.

Then we get the subsequence $\{x^{kk}\}_{k=1}^\infty$ of the sequence $\{x^k\}_{k=1}^\infty$.

By our choice of the sequence $\{x^{kk}\}_{k=1}^\infty$, the sequence $\{x_n^{kk}\}_{k=1}^\infty$ converges, for $n=1, 2, \ldots$. Let a point $x_n^0 \in \mathbf{C}$ be a limit point of the sequence $\{x_n^{kk}\}_{k=1}^\infty$, for $n=1, 2, \ldots$. Let $x^0 = (x_n^0)_{n=1}^\infty$.

We shall verify that the sequence $\{x^{kk}\}_{k=1}^\infty$ converges to the point x^0. For every integer n and every real number $\varepsilon > 0$, there is an integer k_0 such that

$$|x_n^{kk} - x_n^0| < \varepsilon$$

for every $k \geqq k_0$. Therefore we have

$$|x_n^0| \leqq \varepsilon + |x_n^{kk}| \leqq \varepsilon + \varepsilon_n.$$

Since ε is arbitrary, it follows that $|x_n^0| \leqq \varepsilon_n$ for $n=1, 2, \ldots$.

Thus x^0 belongs to the set V. Next, for every positive number δ, there is an integer n_0 such that $\varepsilon_n<\frac{\delta}{2}$ for every $n \geqq n_0$, since $\lim_{n\to\infty} \varepsilon_n=0$. Since $\lim_{k\to\infty} x_n^{kk} = x_n^0$ for $n = 1\ 2,\ldots$, there is an integer k_1 such that

$$| x_n^{kk} - x_n^0 | < \frac{\delta}{2}$$

for $k \geqq k_1$, $n = 1, 2, \ldots, n_0$. Since x^{kk} and x^0 are in V, for $n \geqq n_0$, we have

$$| x_n^{kk} - x_n^0 | \leqq | x_n^{kk} | + | x_n^0 | \leqq \varepsilon_n + \varepsilon_n < \delta$$

for $k = 1, 2, \ldots$. Hence we have

$$\| x^{kk} - x^0 \| = \sup_n | x_n^{kk} - x_n^0 |$$
$$\leqq \max \left(\sup_{1\leqq n\leqq n_0} | x_n^{kk} - x_n^0 |, \sup_{n\geqq n_0} | x_n^{kk}-x_n^0 | \right)$$
$$\leqq \max \left(\frac{\delta}{2}, \delta\right) = \delta$$

for every $k \geqq k_1$. Thus the sequence $\{x^{kk}\}_{k=1}^{\infty}$ converges to x^0.

This implies that V is compact.

Lemma 4. *Let $(\varepsilon_i)_{i\in I} \in C_0(I)$ with $\varepsilon_i \geqq 0$ for $i \in I$. Let $W = \{ (x_i)_{i\in I} \in C_0(I) ;\ | x_i | \leqq \varepsilon_i$ for $i \in I \}$. Then W is a compact subset of $C_0(I)$.*

Proof. Let $J = \{ i \in I ; \varepsilon_i \neq 0 \}$. Then J is a countable subset of I. Hence, we may assume that W is contained in the closed subspace $C_0(J)$ of $C_0(I)$. By Lemma 3, W is a compact subset of $C_0(J)$. Consequently, W is compact subset of $C_0(I)$.

Theorem 5. *If B is a bounding subset of $C_0(I)$, then B is compact.*

Proof. For $j \in I$, a real number $\varepsilon_j \geqq 0$ is defined by

$$\sup \{ | x_j | ;\ (x_i)_{i\in I} \in B \}.$$

Suppose that there are a real number $\delta > 0$ and a countable infinite subset J of I such that $\varepsilon_i \geqq 2\delta$ for $j\in J$. Then, there is a point $x^j=(x^j_i)_{i\in I}\in B$ for $j\in I$ such that $| x^j_j | \geqq\delta$. Let $A = \{ x^j ; j\in J \}$, and $J_1 = \{ j \in I$; there exists a point $(x_i)_{i\in I}$ of A such that $x_j \neq 0\}$. Then J_1 is a countable set. If the set A is an infinite set, A is not a relatively compact subset of $C_0(J_1)$. By S. Dineen [3], the closure $\bar{A}$ of A in $C_0(J_1)$ is not a bounding subset of $C_0(J_1)$. The closed subspaces $C_0(J_1)$, $C_0(I-J_1)$ of $C_0(I)$ are topological supplements. Since B is bounding in $C_0(I)$, $B\cap C_0(J_1)$ is a bounding subset of $C_0(J_1)$. By S. Dineen [3], $B\cap C_0(J_1)$ is a compact subset of $C_0(J_1)$. Since $\bar{A}$ is contained in $B\cap C_0(J_1)$, this contradicts the fact that $\bar{A}$ is non-compact. Thus A is a finite set. Then there are a countable infinite subset J' of J and a point $x= (x_i)_{i\in I}\in A$ such that $| x_j | \geqq \delta$ for $j \in J'$. Then we have $x \notin C_0(I)$. This contradicts $x \in C_0(I)$. Thus $J= \{i\in I ; \varepsilon_i > 0\}$ is a countable subset of I, besides $(\varepsilon_i)_{i\in I}$ belongs to $C_0(I)$. Hence, by Lemma 4 the subset $\{(x_i)_{i\in I} \in C_0(I) ; |x_i|\leqq \varepsilon_i$ for $i \in I\}$ is compact. Since B is contained in this subset, B is compact.

Thus we gain an example of a non-separable and non-reflexive Banach space whose bounding subsets are compact subsets.

Proposition 6. *Let E be a Banach space whose bounding subsets are nowhere dense. Then there exists a bounded sequence of E such that the sequence is not a bounding subset of E.*

Proof. A symbol $\|\cdot\|$ denotes a norm of E. By assumption, the subset $V=\{ x\in E ;\ \| x \| \leqq 1 \}$ is not bounding. Hence there are a holomorphic function $f \in H(E)$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of V such that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is unbounded in C. We can select the sequence $\{x_n\}_{n=1}^{\infty}$ without an accumulation point. Then the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies this proposition.

Finally, we shall discuss a bounding subset of Cartesian products of metrizable locally convex spaces.

THEOREM 7. *Let E and F be metrizable locally convex spaces. If C_E is a bounding subset of E, and C_F is a bounding subset of F, then $C_E \times C_F$ is also a bounding subset of a Cartesian product $E \times F$.*

PROOF. The compact-open topology on the vector space of all continuous functions on F induces a topology τ on $H(F)$. The bornological topology on $H(F)$ associated with τ is denoted by τ_b. Let $f \in H(E \times F)$. We define a mapping $u : E \to H(F)$ by $u(a)(y) = f(a, y)$ for $a \in E$, $y \in F$. By G. Coeuré [1], since E is metrizable, we can verify that the mapping $u : E \to (H(F), \tau)$ is holomorphic. Moreover, since F is metrizable, by [1], u is a holomorphic mapping from into $(H(F), \tau_b)$. Hence the image $u(C_E)$ is a bounded subset of $(H(F), \tau_b)$ by Proposition 2. We define a seminorm on $H(F)$ by

$$p(g) = \|g\|_{C_F} = \sup_{y \in C_F} |g(y)|$$

for $g \in H(F)$. Since F is metrizable, by S. Dineen [2], $(H(F), \tau_b)$ is a barrelled space. For a fixed point y in F, the linear function $g \to g(y)$ of $H(F)$ is continuous with respect to the topology τ_b. Hence, the subset

$$V(y) = \{g \in H(F) ; |g(y)| \leq 1\}$$

of $H(F)$ is a barrel in $(H(F), \tau_b)$. The set

$$B_p = \{g \in H(F) ; p(g) \leq 1\}$$

is absorbing. Thus, since

$$B_p = \bigcap_{y \in C_F} V(y),$$

B_p is a barrel in $(H(F), \tau_b)$. Since $(H(F), \tau_b)$ is barrelled, B_p is a neighborhood of $0 \in H(F)$. Hence p is continuous on $(H(F, \tau_b)$. Hence there is an $M > 0$ such that

$$\sup_{x \in C_E} p(u(x)) \leq M.$$

Since

$$\sup_{x \in C_E} p(u(x)) = \sup_{x \in C_E} \sup_{y \in C_F} |f(x,y)|,$$

we have

$$|f(x, y)| \leq M$$

for all $(x,y) \in C_E \times C_F$. Consequently, $C_E \times C_F$ is a bounding subset of $E \times F$.

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