A Note on M-Saturations I

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Abstract

The purpose of this note is two-fold. First we define, for a fixed module M, the M-saturation of a left Gabriel topology and investigate the relation between localizations determined by the topology and its M-saturation. As a consequent we give a generalization of a result due to Popescu and Spircu [5]. Secondly, in a commutative ring R, we study the relation between the saturation of a given multiplicative system and the R-saturation of the topology corresponding to the original system.

§1. Introduction

First, as preliminaries, we define the radical k_Q for a given module Q. Using this we look at the notion of M-density again.

In the second place, we define the M-saturation of a left Gabriel topology and point out the relation between localizations determined by the topology and its M-saturation.

Finally, in a commutative ring R we study the relation between the saturation of a multiplicative system and the R-saturation in our sense of the topology corresponding to the original system. We provide an example which shows that the former need not imply the latter.

§2. Preliminaries

Let R be a ring with identity. By R-mod we denote the category of unital left R-modules.

For a left R-module Q, let us define

$$k_Q(M) = \cap\{\mathrm{Ker}(f) \mid f \in \mathrm{Hom}_R(M, Q)\}$$

for each R-module M. Then k_Q is a radical of R-mod such that $k_Q(Q)=0$. Moreover it is a unique maximal one of those preradicals r of R-mod for which $r(Q)=0$, and $k_{E(Q)}$ is a unique maximal one of those left exact preradicals r of R-mod for which $r(Q)=0$. The torsion class $\mathrm{T}(k_{E(Q)})$ of $k_{E(Q)}$ is just the class $\{{}_RM \mid \mathrm{Hom}_R(M, E(Q))=0\}$, while the torsion-free class $\mathrm{F}(k_{E(Q)})$ of $k_{E(Q)}$ consists of all left R-modules which can be embedded in a direct product of copies of $E(Q)$.

For a left R-module M, we call a left ideal K of R M-dense, following Shock [6], if $Km \neq 0$ for all $0 \neq m \in E(M)$. It is easy to see that K is M-dense if and only if $\mathrm{Hom}_R(R/K, E(M))=0$, or equivalently, $R/K \in \mathrm{T}(k_{E(M)})$. Hence the set $\{K \subset {}_RR \mid K \text{ is } M\text{-dense}\}$ is nothing but the left Gabriel topology corresponding to $k_{E(M)}$.

For all undefined notions about torsion theories we refer to Stenström [8].

§3. M-saturated Gabriel topologies

Throughout this section we fix a left R-module M. For a submodule M' of M, we denote by $L_{M'}$ the left Gabriel topology consisting of all M/M'-dense left ideals of R, i. e., $L_{M'} = L(k_{E(M/M')})$, where, for a left exact preradical t of R-mod, $L(t)$ means the left linear topology corresponding to t.

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$k_{E(M)}$ is a unique maximal one of those left exact preradicals r of R-mod for which $r(M)=0$. So we obtain

Proposition 1. *Let t be a left exact preradical of R-mod. Then $t(M)=0$ if and only if $L(t)\subset L(k_{E(M)})$.*

From this proposition we have the following Spulber's results.

Corollary 2 ([7, **Proposition 3.6**]) . *For any left exact radical t of R-mod we have $L(t)\subset L_{t(M)}$.*

Proof. Since $t(M/t(M))=0$, Proposition 1 says that $L(t)\subset L(k_{E(M/t(M))})=L_{t(M)}$.

Corollary 3 ([7, **Proposition 3. 7**]) . For a nonsingular module M each essential left ideal of R is M-dense.

Proof. For the left exact preradical Z, the corresponding left linear topology $L(Z)$ is $\{K\subset {}_RR \mid K$ is essential in $R\}$. So if K is essential in R, then $K\in L(k_{E(M)})$ by Proposition 1. Thus K is M-dense.

Definition ([7]). A left Gabriel topology L on R is M-saturated if for any left Gabriel topology L' such that $t(M)=t'(M)$, $L'\subset L$ holds, where t, t' are left exact radicals of R-mod corresponding to L, L' respectively.

Spulber gives the following characterization of this concept.

Theorem 4 ([7, **Theorem 4.1**]) . *For a left exact radical t of R-mod, the following statements are equivalent*:

(a) *$L(t)$ is M-saturated.*

(b) *$L(t)=L_{t(M)}$.*

Proof. (a) $\Longrightarrow$ (b). Take any $x\in k_{E(M/t(M))}(M)$. For the canonical R-homomorphism π from M to $M/t(M)$, we have $\pi(x)=0$. So $x\in t(M)$ and hence $k_{E(M/t(M))}(M)\subset t(M)$. On the other hand, according to Corollary 2, $L(t)\subset L_{t(M)}$, i. e., $t\leq k_{E(M/t(M))}$. Hence we have $k_{E(M/t(M))}(M)=t(M)$. Since $L_{t(M)}\subset L(t)$ by the definition of M-saturatedness, we see that $L_{t(M)}=L(t)$.

(b)$\Longrightarrow$(a). Lat L' be a left Gabriel topology with left exact radical t' such that $t(M)=t'(M)$. Corollary 2 implies that $L'\subset L_{t'(M)}=L_{t(M)}=L(t)$. This means that $L(t)$ is M-saturated.

Corollary 5. *For a left Gabriel topology L with left exact radical t there exists a unique M-saturated left Gabriel topology L' with left exact radical t' such that $L\subset L'$ and $t(M)=t'(M)$.*

Proof. We put $L'=L_{t(M}$. By Corollary 2, we have $L\subset L'$ and by the proof of Thorem 4 L' is M-saturated and $t(M)=t'(M)$ folds. To prove the uniqueness, we assume that $L''$is an M-saturated left Gabriel topology with left exact radical t'' such that $L\subset L''$ and $t(M)=t''(M)$. Since L' is M-saturated with $t'(M)=t''(M)$, then $L''\subset L'$ and since L'' is M-saturated with $t'(M)=t''(M)$, then $L'\subset L''$. Hence we conclude that $L'=L''$.

Definition. We call the M-saturated left Gabriel topology L' in Corollary 5 the M-saturation of L.

Proposition 6. *Let L be a left Gabriel topology with left exact radical t and L' its M-saturation. Then $t(M)=0$ if and only if $L'=\{K\subset {}_RR \mid K$ is M-dense$\}$.*

Proof. If $t(M)=0$, then $L'=L_{t(M)}=L(k_{E(M)})$. Conversely if $L'=L(k_{E(M)})$, then $t(M)=t'(M)=k_{E(M)}(M)=0$.

For example if $L\subset\{K\subset{}_RR \mid K$ is M-dense$\}$, then $t(M)\subset k_{E(M)}(M)=0$. So $\{K\subset {}_RR \mid K$ is M-dense$\}$ is just the M-saturation of L.

Theorem 7. *Let L be a left Gabriel topology with left exact radical t and L' its M-saturation. Then the module of quotients $M_{L'}$ of M with respect to L' contains M_L, that of M with respect to L, and $M_L=\{x\in M_{L'} \mid ((M/t(M)):x)\in L\}$ holds.*

Proof. It is well known that the module of quotients M_L of M with respect to L is expressed as $M_L=\{x\in E(M/t(M)) \mid ((M/t(M)):x)\in L\}$. Let t' be the left exact radical corresponding to L'. Since

$t(M)=t'(M)$ and $L \subset L'$, we can see that $M_L \subset M_{L'}$ and $M_L=\{x \in M_{L'} \mid ((M/t(M)) : x) \in L\}$.

Corollary 8 ([5, pp. 44—45]). *Let L be a left Gabriel topology with left exact radical t and L′ its R-saturation. Then $R_{L'}$ is isomorphic to the double commutator R'' of $E(R/t(R))$ and R_L is isomorphic to $\{f \in R'' \mid ((R/t(R)) : f) \in L\}$.*

Corollary 9. *Let L be a left Gabriel topology with left exact radical t and L′ its M-saturation. If $t(R)=R$, then the module of quotients M_L of M with respect to L coincides with $M_{L'}$, that of M with respect to L′.*

Proof. By Corollary 2 and Theorem 4, we see that $L'=L_{t(M)}$ and $L \subset L'$. Since R is torsion, L consists of all left ideals of R. Therefore we have $L=L'$ and $M_L = M_{L'}$.

§4. Saturations of multiplicative systems

Throughout this section we assume that R is commutative.

Definition. A subset S of R is called a multiplicative system in R if

(1) $a, b \in S \Longrightarrow ab \in S$, and

(2) $0 \notin S$.

Proposition 10. *If S is a multiplicative system in R, then $L=\{K \subset {}_RR \mid K \cap S \neq \phi\}$ is a left Gabriel topology.*

Proof. Clear.

Definition ([4]). For a multiplicative system S in R, the set $\overline{S}=\{x \in R \mid xy \in S$ for some $y \in R\}$ is the saturation of S.

Definition ([1]). A multiplicative system S in R is said to be saturated if $ab \in S$, then $a \in S$ and $b \in S$.

Lemma 11. *The saturation $\overline{S}$ of a multiplicative system S is a smallest saturated multiplicative system containing S.*

Proof. Obviously $\overline{S}$ is a multiplicative system containing S. We shall point out that $\overline{S}$ is saturated. We assume that either $a \notin \overline{S}$ or $b \notin \overline{S}$. If $a \notin \overline{S}$, then by definition $ax \notin S$ for all $x \in R$. So we have $(ab)c = a(bc) \notin S$ for all $c \in R$ and thus $ab \notin \overline{S}$. Next let S' be a saturated multiplicative system containing S. For any $a \in \overline{S}$, there exists $b \in R$ such that $ab \in S \subset S'$. Since S' is saturated, a is in S'. This implies that $\overline{S} \subset S'$. Therefore, $\overline{S}$ is a smallest saturated multiplicative system containing S.

Proposition 12. *Let S be a multiplicative system in R. If S is contained in the set T of all regular elements of R, then*

(1) *the saturation $\overline{S}$ of S is also contained in T, and*

(2) *any ideal K of R with $K \cap S \neq \phi$ is R-dense.*

Proof. (1) The set T is a saturated multiplicative system containing S. So the proof of (1) follows from Lemma 11.

(2) Take s in $K \cap S$. For any nonzero element x in $E(R)$ we have a nonzero element ax in $Rx \cap R$ for some a in R. Since $ax \neq 0$ and s is regular, $0 \neq s(ax)=(sa)x \in Kx$. So K is R-dense.

Using this Proposition we see that the Gabriel topology $L=\{K \subset R \mid K \cap S \neq \phi\}$ corresponding to S is contained in $\{K \mid K$ is an R-dense ideal$\}$ and by Proposition 6 the R-saturation L' of L is precisely the set $\{K \mid K$ is an R-dense ideal$\}$.

Corollary 13. *For a domain R, an ideal K of R is R-dense if and only if $K \neq 0$.*

Proof. Clear.

Proposition 14. *Let S be a multiplicative system in R and $\bar{S}$ its saturation. Then the Gabriel topology $L=\{K\subset R \mid K\cap S\neq\phi\}$ corresponding to S coincides with the Gabriel topology $\bar{L}=\{K\subset R \mid K\cap\bar{S}\neq\phi\}$ corresponding to $\bar{S}$.*

Proof. Let K be an ideal in $\bar{L}$, and take a nonzero element a in $K\cap\bar{S}$. We can fined $b\in R$ such that $ab\in S$. The element ab is in K and hence $K\cap S\neq\phi$. Thus we have $\bar{L}\subset L$.

Frome now on, we assume that R is a commutative Noetherian ring. In this case, according to Stenström [8], for every Gabriel topology L there is a subset P of Spec (R) such that $L=\{K \mid V(K)\cap P=\phi\}$, where $V(K)=\{p\in \text{Spec }(R) \mid K\subset p\}$.

Theorem 15. *Let S be a multiplicative system in R and $\bar{S}$ its saturation. We denote by L and $\bar{L}$ the corresponding Gabriel topologies as above. We express the R-saturation L' of L as $L'=\{K \mid V(K)\cap P=\phi\}$ for some $P\subset$ Spec (R). Then*

(1) *$S'=R-\cup P$ is a saturated multiplicative system containing $\bar{S}$.*

(2) *The Gabriel topology $L''=\{K\subset R \mid K\cap S'\neq\phi\}$ corresponding to S' is contained in L'.*

(3) *If the following condition (*) is satisfied, then $L'\subset L''$ and thus we have $L'=L''$.*

(*) *If $K\subset\cup P$, then $K\subset p$ for some $p\in P$.*

Remark. If P is finite, then the condition (*) is satisfied as is well-known.

Proof. (1) We only show that $\bar{S}\subset S'$. Let s be any element in $\bar{S}$. If s is in $\cup P$, then there is a prime ideal p in P such that $s\in p$ and so $p\cap\bar{S}\neq\phi$. This implies that $p\in\bar{L}=L\subset L'$ and $V(p)\cap P=\phi$. But this is a contradiction. Thus $s\notin\cup P$.

(2) Let K be an ideal of R such that $K\cap S'\neq\phi$. There is an element s in $K\cap S'$. Since every ideal of $V(K)$ contains s and any ideal of P dose not contain s, $V(K)\cap P=\phi$. Thus we have $K\in L'$.

(3) Let K be an arbitrary element in L'. Then $V(K)\cap P=\phi$. Consequently if p is in P, then p is not in $V(K)$ and hence K is not contained in p. By the condition (*), K is not contained in $\cup P$. Therefore, there exists an element a in K such that $a\notin\cup P$. thus $a\in K\cap S'$, which shows that $K\in L''$.

Example 16. Let R be the ring of integers. The subset $S=\{1\}$ of R is a multiplicative system and its saturation $\bar{S}$ is the set $\{1,-1\}$. The Gabriel topologies L and $\bar{L}$ corresponding to S and $\bar{S}$ coincide and are equal to $\{R\}$. The R-saturation L' of L is of the form $\{K \mid K$ is an R-dense ideal$\}=\{K \mid K\neq 0\}$ and is also expressed by $F_P=\{K \mid V(K)\cap P=\phi\}$ for some P. However we can claim that $P=\{0\}$. For if there exists $(p)\neq 0$ in P, then (p) is in L'. Hence we have $V((p))\cap P=\phi$, a contradiction. As in Theorem 15, we make the multiplicative system S', i. e., $S'=R-\cup P=R-(0)$. Surely $S\subsetneqq\bar{S}\subsetneqq S'$, but P is finite and so $L'=L''$ by Theorem 15 (3), where L'' is the Gabriel topology corresponding to S'.

After all we can conclude that the Gabriel topology corresponding to the saturation of a multiplicative system need not coincide in general with our R-saturation of the Gabriel topology corresponding to the original multiplicative system.

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