Essential Extensions of Modules in a Torsion Theory

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Abstract

Our purpose of this note is, using a concept of σ-essential extensions, to study some basic properties concerning essential extensions of modules in a torsion theory.

Introduction

For a fixed left exact radical σ on *R-mod*, a module M is called a σ-essential extension of a submodule N, if M is essential over N and $M/N \in \mathbf{T}$. In this paper we study some basic properties concerning σ-essential extensions of modules. We treat σ-uniform modules, σ-atomic modules and σ-critical left ideals.

Essential extensions of modules

Throughout this paper, R will mean a ring with identity 1 and modules will mean unital left R-modules.

We denote by *R-mod* the category of all modules.

For a left exact radical σ on *R-mod*, we associate a hereditary torsion theory $(\mathbf{T}, \mathbf{F})$ for *R-mod*, and an idempotent filter $\mathcal{L}$.

Basic properties of torsion theories used in this paper may be found in [1].

A module M is said to be a *σ-essential extension* of a submodule N if,

(1) M is an essential extension of N, and

(2) N is σ-open in M, (i.e., $M/N \in \mathbf{T}$).

A non-zero module M is said to be *σ-uniform* if M is a σ-essential extension of every non-zero submodule.

Lemma 1. *Let L, M and N be modules with $L \subset M \subset N$. If M is a σ-essential extension of L and N is a σ-essential extension of M, then N is a σ-essential extension of L, and conversely.*

Proof. Assume that M and N are σ-essential extensions of L and M respectively. Then it is clear that N is an essential extension of L. From the exact sequence $0 \longrightarrow M/L \longrightarrow N/L \longrightarrow N/M \longrightarrow 0$, it follows that $N/L \in \mathbf{T}$, i.e., L is σ-open in N.

Conversely, assume that N is a σ-essential extension of L. It is clear that M is an essential extension of L and N is an essential extension of M. Since the sequences $0 \longrightarrow M/L \longrightarrow N/L$ and $N/L \longrightarrow N/M \longrightarrow 0$ are exact, we have $M/L \in \mathbf{T}$ and $N/M \in \mathbf{T}$.

From the above lemma, we see that every non-zero submodule and every σ-essential extension of a σ-uniform module are again σ-uniform.

A module is said to be *σ-complete*, if it has no proper σ-essential extensions.

Proposition 2. *For every module M, there is a σ-essential extension $\bar{M}$ of M which is σ-complete. This extension is unique up to isomorphism over M. $\bar{M}$ is called the σ-completion of M.*

Proof. We can construct an extension $\bar{M}$ of M by $\bar{M}/M = \sigma\,(E(M)/M)$, where $E(M)$ denotes the injective hull of M. Since $M \subset \bar{M} \subset E(M)$ and $E(M)$ is an essential extension of M, $\bar{M}$ is an essential extension of M, and moreover, $\bar{M}/M$ is in $\mathbf{T}$. This means that $\bar{M}$ is a σ-essential extension of M.

Next, we shall show that $\bar{M}$ is σ-complete. Let H be a σ-essential extension of $\bar{M}$. We may assume that H is contained in $E(M)$. Since $H/M \in \mathbf{T}$, we obtain that $H/M \subset \sigma\,(E(M)/M) = \bar{M}/M$, so that $\bar{M}=H$.

It remains to show that $\bar{M}$ is unique. To this end, assume that U is any σ-essential extension of M and is σ-complete. Since $M \subset U \subset E(M)$ and $U/M \in \mathbf{T}$, we have $U/M \subset \bar{M}/M$, i.e., $U \subset \bar{M}$. Using Lemma 1, $\bar{M}$ is a σ-essential extension of U. Thus $U = \bar{M}$ because of the σ-completeness of U.

A module E is called *σ-injective* if for any $I \in \mathcal{L}$ and any R-homomorphism f of I to M, there is an R-homomorphism $\bar{f}$ of R to M extending f.

Proposition 3. *A module M is σ-complete if and only if M is σ-injective.*

Proof. Let M **be** σ-injective and let M' be a σ-essential extension of M. The sequence $0 \longrightarrow M \longrightarrow M' \longrightarrow M'/M \longrightarrow 0$ is exact and $M'/M \in \mathbf{T}$. Hence there exists some module X such that $M' = M \oplus X$. But M' is an essential extension of M and so we obtain $M' = M$.

Conversely, let $I \in \mathcal{L}$ and let f be an R-homomorphism of I to M. By the injectivity of $E(M)$, there exists an R-homomorphism $\bar{f}$ of R to $E(M)$ extending f. We put $\bar{f}(1)=x$ and show that $x \in M$. Let φ be the mapping from R to $(Rx+M)/M$ defined by $\varphi(r) = rx + M$ for $r \in R$. Then $\ker \varphi = (M : x)$ and $R/\ker \varphi \cong (Rx+M)/M$. For any $a \in I$, $((M : x) : a) = (M : ax) = R \in \mathcal{L}$, since $ax = \bar{f}(a) = f(a) \in M$. By the property of $\mathcal{L}$ in [1, P, 7] we have $(M : x) \in \mathcal{L}$. Thus $(Rx+M)/M \in \mathbf{T}$, and so $Rx+M$ is a σ-essential extension of M. By assumption $M = Rx+M$. Therefore $x \in M$.

Proposition 4. *If M is σ-injective and E(M) is σ-uniform, then M is injective.*

Proof. *Since* M is σ-injective, $E(M)/M \in \mathbf{F}$ (see Lambek [1]). On the othere hand, since $E(M)$ is σ-uniform, we have $E(M)/M \in \mathbf{T}$. Thus $E(M) = M$ as desired.

Proposition 5. *Let M be a uniform module. Then M is* **σ**-*uniform if and only if, for any non-zero element m of M, Rm is σ-uniform.*

Proof. The "only if" part is obvious. To prove the "if" part, assume that M is not σ-uniform. Then there is some non-zero submodule N of M such that M is not a σ-essential extension of N. Since M is uniform, N is not σ-open in M. There exists a proper submodule L of M such that $\sigma(M/N) = L/N$, and $0 \neq M/N \cong (M/N)/\sigma(M/N) \in \mathbf{F}$. Take an element m in $M \backslash L$. Since $Rm/(Rm \cap L)$ is isomorphic to a submodule of M/L, $Rm/(Rm \cap L) \in \mathbf{F}$. On the other hand, M is uniform and $L \neq 0$, so we have $Rm \cap L \neq 0$.

Hence $Rm/(Rm \cap L) \in \mathbf{T}$ by assumption. This implies that $Rm/(Rm \cap L) \in \mathbf{T} \cap \mathbf{F} = 0$ and $Rm = Rm \cap L$. Thus $m \in L$, a contradiction.

A left ideal I of R is called *σ-critical*, if R/I is σ-uniform. Thus I is σ-critical if and only if R/I is an essential extension of J/I and $R/J \in \mathbf{T}$, for all left ideals J properly containing I.

A module M is called *σ-atomic* if

(1) M is σ-uniform and

(2) M is σ-complete.

By the above definition, we have the following proposition.

Proposition 6. *If M is σ-uniform, then* $\bar{M}$ *is σ-atomic. In particular, if I is a σ-critical left ideal of R, then* $\overline{(R/I)}$ *is σ-atomic.*

Moreover, we can prove the converse of this proposition.

Proposition 7. *Let M be a σ-atomic module. Then there exists a σ-critical left ideal I of R such that M is isomorphic to $\overline{(R/I)}$.*

Proof. For any m $(\neq 0)$ in M, let $I=(0:m)$. Then R/I is isomorphic to Rm. Since M is σ-uniform, Rm is σ-uniform. Thus, I is σ-critical. Moreover M is a σ-essential extension of Rm. Hence we have $M \cong \overline{(R/I)}$.

Proposition 8. *The following statements are equivalent for a non-zero module M:*

(a) *M is σ-uniform.*

(b) *$\overline{M}$ is σ-uniform.*

(c) *$\overline{M}$ is σ-atomic.*

Proof. Obvious.

Proposition 9. *The following statements are equivalent for two σ-critical left ideals I and J:*

(a) *I and J are related (i.e., if $(I:a)=(J:b)$ for some $a \in R\backslash I$ and $b \in R\backslash J$).*

(b) *A non-zero submodule of R/I is isomorphic to a submodule of R/J.*

(c) $\overline{(R/I)} \cong \overline{(R/J)}$.

Proof. The proof is similar to that of Storrer [2, Proposition 2. 3], so we omit the proof.

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References

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2) H. Storrer, On Goldman's primary decomposition, Lecture Notes in Math. No. 246. Springer (1972), 617—661.

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