On a generalization of Papp's theorem

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Abstract

Recently, using a concept of S-modules, Z. Papp has given some equivalent conditions for a left Artinian ring to be hereditary. But, apart from this concept, we can prove the equivalence of these conditions in the point of view of general 3-fold torsion theories. This is our purpose of this note.

§1. Introduction

An R-module M is called an S-module if every homomorphic image of its injective hull is injective. As was pointed out by Papp, if R is left Noetherian, the class **S** of all S-modules forms a torsion class, that is, **S** is closed under homomorphic images, direct sums and group extensions. Moreover **S** is stable and hereditary in the sense that it is closed under injective hulls and submodules.

The class of R-modules having no S-submodule other than 0 forms the associated torsion-free class **F** of **S**.

In particular, if R is left Artinian, the associated filter of left ideals of R determined by the torsion theory (**S**, **F**) has a minimal element. Therefore **S** becomes a TTF-class and together with the class **C** of those R-modules whose non-zero homomorphic image is not S-module, the triple (**C**, **S**, **F**) forms a 3-fold torsion theory for R-*mod*. For this 3-fold torsion theory, Papp has shown that the following six conditions are equivalent:

(a) *R is a hereditary ring.*

(b) *R/N is an S-module.*

(c) *Every simple R-module is an S-module.*

(d) $c(R) = 0$.

(e) $s(R) = R$.

(f) *All R-modules are S-modules.*

The purpose of this note is to show the above equivalences except for (a) from a point of view of general 3-fold torsion theories.

§2. Preliminaries

Throughout this paper, R will mean a ring with identity and R-modules will mean unital left R-modules.

Following Dickson [1], we shall make definitions:

A *torsion theory* for R-*mod*, the category of left R-modules, consists of a couple (**T**, **F**) of classes of R-modules satisfying the following axioms:

(1) $\mathbf{T} \cap \mathbf{F} = \{0\}$.

(2) If $T \to A \to 0$ is exact with $T \in \mathbf{T}$ then $A \in \mathbf{T}$.

(3) If $0 \to A \to F$ is exact with $F \in \mathbf{F}$ then $A \in \mathbf{F}$.

(4) For each R-module M, there exists a submodule $t(M)$ of M such that $t(M) \in \mathbf{T}$ and $M/t(M)$

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$\in$ **F**.

The modules in **T** are torsion modules and those in **F** are torsion free and $t(M)$ is the unique largest submodule of M in **T**.

Let **T** be a class of R-modules. Then **T** is a torsion class if there exists a class **F** such that (**T** , **F**) forms a torsion theory. A torsion-free class is defined dually. A torsion class **T**, and the associated torsion theory (**T** , **F**), is called hereditary (stable) if **T** is closed under submodules (injective hulls). Note that if **T** is hereditary, then **T** stable means that **T** is closed under essential extensions.

The following results are due to Dickson [1].

(1 . 1) A class **T** of R-modules is a torsion class if and only if **T** is closed under homomorphic images, arbitrary direct sums, and extensions. Dually, a class **F** is a torsion-free class if and only if **F** is closed under submodules, arbitrary direct products, and extensions.

(1 . 2) Let (**T** , **F**) be a torsion theory. Then **T** and **F** uniquely determine each other as follows:

$\mathbf{T}=\{M\in R\text{-}mod \mid \mathrm{Hom}_R(M, N)=0 \text{ for all } N\in \mathbf{F}\}$,

$\mathbf{F}=\{M\in R\text{-}mod \mid \mathrm{Hom}_R(N, M)=0 \text{ for all } N\in \mathbf{T}\}$.

(1 . 3) If (**T** , **F**) is a torsion theory, then **T** is hereditary if and only if **F** is closed under injective hulls.

In [3], Kurata has defined an n-fold torsion theory for *R-mod* as follows.

For any integer $n>1$, an *n-fold torsion theory* for *R-mod* consists of an n-tuple

$$(\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n)$$

of classes of R-modules such that each successive pair $(\mathbf{T}_i, \mathbf{T}_{i+1})$, for $i=1, 2, \ldots, n-1$, forms a torsion theory. Now, let $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$ be a 3-fold torsion theory. This is nothing but a TTF-theory defined by Jans [2]. This means that $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}_2, \mathbf{T}_3)$ are torsion theories with torsion radicals t_1 and t_2 respectively.

§ 3. C is hereditary

In [4], Papp has proved that the class **F** contains the class **C**. But this is equivalent to the fact that **C** is closed under submodules, that is, **C** is hereditary, and this is certainly true by (1 . 3) since **S** is stable.

Proposition. $\mathbf{F}\supset\mathbf{C}$ *if and only if* **C** *is closed under submodules.*

Proof. By $s(M)$ and $c(M)$ we shall denote the torsion submodules of an R-module M with respect to (**S** , **F**) and (**C** , **S**) respectively. The "if" part was proved in Lemma 2 . 2 of Kurata [3]. To prove the "only if" part we need a well-known lemma:

Lemma. **C** *is closed under submodules if and only if* $N\subset M$ *then* $c(N)=c(M)\cap N$ *for all* $M, N\in R$-*mod*.

Proof. The "if" part is clear. Clearly $c(N)$ is contained in $c(M)\cap N$. Conversely, if m is an element of $c(M)\cap N$, then $Rm\subset c(M)$ and $Rm\subset N$. Since **C** is closed under submodules, Rm belongs to **C** and hence we obtain that $Rm\subset c(N)$. Thus $m\in c(N)$. This establishes the lemma.

Proof of the "only if" part of Proposition. The following proof is due to Kurata. Let M be an R-module and N its submodule. It is clear that $c(N)$ is contained in $c(M)\cap N$. Since $\frac{c(M)\cap N}{c(c(M)\cap N)}$ is in **S**, we have that $\frac{c(M)\cap N}{c(c(M)\cap N)}\subset s\left(\frac{c(M)}{c(c(M)\cap N)}\right)\subset\frac{c(M)}{c(c(M)\cap N)}$, and since $\frac{c(M)}{c(c(M)\cap N)}$ is in **C**, we have by assumption that $s\left(\frac{c(M)}{c(c(M)\cap N)}\right)\in\mathbf{S}\cap\mathbf{F}=0$. Hence $c(M)\cap N=c(c(M)\cap N)$. This means that $c(M) \cap N$ is in **C** and thus $c(M) \cap N \subset c(N)$. This completes the proof of Proposition.

§ 4. Main theorem

We are now ready to prove the following theorem. This is our main theorem.

Theorem. *Let R be a ring with identity and N its Jacobson radical. For any 3-fold torsion theory* ($\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$), *we consider the following conditions*:

(1) $t_1(R) \subset N$.

(2) *R/N is in* $\mathbf{T}_2$.

(3) *Every simple R-module is in* $\mathbf{T}_2$.

(4) *Every cyclic R-module is in* $\mathbf{T}_2$.

(5) *Every finitely generated R-module is in* $\mathbf{T}_2$.

(6) $t_1(R)=0$.

(7) $t_2(R)=R$.

(8) $\mathbf{T}_2=R$-*mod*.

(9) $\mathbf{T}_1=0$.

(10) *Every projective R-module is in* $\mathbf{T}_2$.

Then, (1)—(3) and (5)—(10) are equivalent, and the implications (5) → (4) → (3) are also true. Moreover if either $\mathbf{T}_1 \subset \mathbf{T}_3$ or $\mathbf{T}_3 \subset \mathbf{T}_1$, then (3) → (6) or (3) → (7) is true and hence all conditions are equivalent.

Proof. (5) → (4) → (3) are obvious and it is not hard to show that (5)—(10) are equivalent.

(1) → (2). Since $R/t_1(R) \to R/N \to 0$ is exact and since $\mathbf{T}_2$ is closed under homomorphic images, R/N is in $\mathbf{T}_2$.

(2) → (3). Every simple R-module is of the form R/I where I is a maximal left ideal of R. Since the Jacobson radical N of R is contained in I, we can show that R/I is in $\mathbf{T}_2$ just like the proof of (1) → (2) above.

(3) → (1). By Proposition 2.4 of Dickson [1], we have $t_1(R)=\cap\{I \mid R/I \in \mathbf{T}_2\}$, where I is a left ideal of R. For any maximal left ideal M of R, R/M is in $\mathbf{T}_2$ by assumption, and so $t_1(R)$ is contained in M. Thus we have $t_1(R) \subset N$.

(3) → (6). Assume that $\mathbf{T}_1 \subset \mathbf{T}_3$, that is, $\mathbf{T}_1$ is closed under submodules by Proposition. We shall claim $t_1(R)=0$. If not, we can find an element $x(\neq 0) \in t_1(R)$. There exists a simple R-module M such that $Rx \to M \to 0$ is exact. Since $Rx \in \mathbf{T}_1$, M is in $\mathbf{T}_1$ and hence M is in $\mathbf{T}_3$ again by assumption. So we have $M \in \mathbf{T}_2 \cap \mathbf{T}_3 = 0$, a contradiction.

(3) → (7). Assume that $\mathbf{T}_3 \subset \mathbf{T}_1$, that is, $\mathbf{T}_3$ is closed under homomorphic images. This fact is proved as follows: $\mathbf{T}_3 \subset \mathbf{T}_1$ means that $R=t_1(R)+t_2(R)$. (see [3], P. 564.) It follows from this $t_2(R)$ is an idempotent two-sided ideal in R and hence $\mathbf{T}_2=\{M \in R\text{-}mod \mid t_1(R)\cdot M=0\}=\{M \in R\text{-}mod \mid t_2(R)\cdot M=M\}$. So we have, by (1.2), $\mathbf{T}_3=\{M \in R\text{-}mod \mid t_2(R)\cdot M=0\}$, which means that $\mathbf{T}_3$ is also a TTF-class.

Now we shall return to the proof of (3) → (7). The following proof is due to Katayama. Suppose that $t_2(R) \neq R$. Then there exists a simple R-module M such that $R/t_2(R) \to M \to 0$ is exact. Since $R/t_2(R)$ is in $\mathbf{T}_3$, M is in $\mathbf{T}_3$ and hence $M \in \mathbf{T}_2 \cap \mathbf{T}_3 = 0$, a contradiction. This completes the proof of the theorem.

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References

1) S. E. Dickson, A torsion theory for abelian categories, Trans. Amer. Math. Soc. 121 (1966), 223—235.

2) J. P. Jans, Some aspects of torsion, Pacific J. Math. 15 (1965), 1249—1259.

3) Y. Kurata, On an n-Fold Torsion Theory in the Category $_{R}M$, J. Algebra 22 (1972), 559—572.

4) Z. Papp, S-modules and Torsion Theories over Artinian Rings, Arch. Math. 23 (1972), 598—602.

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