THE NECESSARY CONDITIONS OF THE SADDLE POINT IN THE THEORY OF DIFFERENTIAL GAME.

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Abstract

This article deals with a differential game with fixed time interval. In the the case that the game has a saddle point, the necessary conditions, which the saddle point must satisfy, are formulated.

1. Introduction. In this article, we shall specialize the general theory of saddle point in (Ref. 1) to a differential game with fixed time interval.

In section 2, we shall introduce a certain class of, two-person differential game, and some definitions. In section 3, we shall introduce some assumptions for the differential system, and assuming that the differential game has the saddle point, we shall show that the saddle point is transformed to (G_1, G_2, A_1, A_2)-saddle point obtained in (Ref. 1). In section 4, we shall formulate the necessary conditions which the saddle point must satisfy, and give its proof.

2. Formulation of the Differential Game and Saddle Point. In this section, we shall formulate a certain class of, two-person differential game, the so called game of degree, and its saddle point.

Let G be an open set in R^n (an n-dimensional linear vector space), let U and V be arbitrary (but fixed) sets in R^r and R^s, respectively, and let I be a bounded open time interval. Let Ω_u (or Ω_v) be set of all functions defined on I which are measurable[1)], essentially bounded, whose range are contained in U (or V), and let J be a closed time interval such that

$$J=[\tau_1, \tau_2] \subset I.$$

In order to define the differential game, there must given:

(i) a continuous function $f(x, u, v, t)$ from $G\times U\times V\times I$ into R^n and a real valued continuous function $g(x, u, v, t)$ from $G\times U\times V\times I$ into R^1 which are of class C^1 with respect to $x\in G$ and measurable in (u, v, t) for every fixed $x\in G$,

(ii) a real valued continuous functions $\alpha^i(x)$, $i=1, \cdots\cdots, m$, $\beta^j(x)$, $j=1, \cdots\cdots, \ell$, $(m+1\leqq n)$, and $\gamma(x)$ defined on G and of class C^1 with respect to $x\in G$.

Now, we can formulate the differential game.

Player P 1 wants to take his strategy $u(t)\in\Omega_u$ such that $u(t)$ and $x(t)$ satisfy

$$\frac{dx}{dt}=f(x, u, v, t),\ x(\tau_1)=c_0,\quad t\in I \tag{2・1}$$

$$\alpha^i(x(\tau_2))\leqq 0,\qquad i=1, \cdots\cdots\cdots, m, \tag{2・2}$$

minimize the payoff function $P(u, v)$ defined by

$$P(u, v)=\gamma(x(\tau_2))+\int_{\tau_1}^{\tau_2} g(x, u, v, t)dt, \tag{2・3}$$

and player P 2 wants to take his strategy $v(t)\in\Omega_v$ such that $v(t)$ and $x(t)$ satisfy (2・1) and

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1) In this paper, measurability is to be understood in the following sence: a function is measurable if the preimage of every Borel set is a Borel set.

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$\beta^j(x(\tau_2)) \leqq 0, \qquad j=1, \cdots\cdots, \ell$ (2・4)

maximize the payoff function $P(u, v)$ defined by (2・3), where $c_0 \in G$ is the initial state of the game.

In this paper, let us use the words "optimal strategy" and "saddle point" as following sence.

Definition. 2・1. (1) *The strategy $\bar{u} \in \Omega_u$ is the optimal strategy of player P 1, if $\bar{u}$ satisfies (2・1) and (2・2) for any $v \in \Omega_v$, and satisfies the relation*

$$\max_{v \in \Omega_v} P(\bar{u}, v) = \min_{u \in \Omega_u} \max_{v \in \Omega_v} P(u, v), \tag{2・5}$$

(2) *The strategy $\bar{v} \in \Omega_v$ is the optimal strategy of player P 2, if $\bar{v}$ satisfies (2・1) and (2・4) for any $u \in \Omega_u$, and satisfies the relation*

$$\min_{u \in \Omega_u} P(u, \bar{v}) = \max_{v \in \Omega_v} \min_{u \in \Omega_u} P(u, v). \tag{2・6}$$

Definition. 2・2. *If the optimal strategies $\bar{u}$ and $\bar{v}$ of both players satisfy the relation*

$$P(\bar{u}, v) \leqq P(\bar{u}, \bar{v}) \leqq P(u, \bar{v}), \quad \text{for every} \quad u \in \Omega_u, \ v \in \Omega_v,$$

then the pair of strategies $(\bar{u}, \bar{v})$ is called the saddle point of the game, and the trajectory $\bar{x}(t)$ corresponding to $(\bar{u}, \bar{v})$ is called optimal trajectory.

3. Assumptions for the Differential System and Preliminary Results.

In this section, in order to obtain a meaningful necessary conditions for the saddle point, we shall introduce some assumptions for the differential system. And we shall transform the differential game into a game in a real Banach spase.

Assumption. *For every functions $u(t) \in \Omega_u$ and $v(t) \in \Omega_v$, and every compact set X of G, there exist functions $m_1(t)$ and $m_2(t)$, integrable over I and possibly depending on $u(t)$, $v(t)$ and X, such that*

$$|f(x, u(t), v(t), t)| \leqq m_1(t), \quad \left| \frac{\partial f(x, u(t), v(t), t)}{\partial x} \right| \leqq m_1(t),$$

$$|g(x, u(t), v(t), t)| \leqq m_2(t), \quad \left| \frac{\partial g(x, u(t), v(t), t)}{\partial x} \right| \leqq m_2(t),$$

for every $x \in X$ and $t \in I$.

Here, vertical bars denote any vector norm in a finite dimensional linear vector space.

Let $x^0(t)$, $t \in I$ be the function of t satisfied the differential equation

$$\frac{dx^0(t)}{dt} = g(x(t), u(t), v(t), t), \quad x^0(\tau_2) = 0, \quad t \in I. \tag{3・1}$$

If $x^0(t)$ and $x(t)$ satisfy the differential equations (3・1) and (2・1), respectively, then the $n+1$ dimensional vector valued function

$$z(t) = \begin{pmatrix} x^0(t) \\ x(t) \end{pmatrix}, \quad t \in I,$$

satisfies the differential equation

$$\frac{dz(t)}{dt} = F(z(t), u(t), v(t), t), \quad z(\tau_1) = z_0, \ t \in I, \tag{3・2}$$

where,

$$F(z(t), u(t), v(t), t) = \begin{pmatrix} g(x(t), & u(t), & v(t), & t) \\ f(x(t), & u(t), & v(t), & t) \end{pmatrix}, \quad z_0 = \begin{pmatrix} 0 \\ c_0 \end{pmatrix}.$$

By assumption, the following lemma evidently holds.

Lemma. 3・1 *For every functions $u(t)\in \Omega_u$ and $v(t)\in \Omega_v$, and every compact set Z of $R^1\times G$, there exists a function $m(t)$, integrable over I and possibly depending on $u(t)$, $v(t)$, and Z, such that*

$$| F(z,\ u(t),\ v(t),\ t) | \leqq m(t), \quad \left| \frac{\partial F(z,\ u(t),\ v(t),\ t)}{\partial z} \right| \leqq m(t),$$

for every $z \in Z$ and $t \in I$.

Let $\xi^i(z)$, $i=1, \cdots\cdots, m$ and $\zeta^j(z)$, $j=1, \cdots\cdots, \ell$ be a functions defined by

$$\xi^i(z)=\alpha^i(x), \qquad i=1, \cdots\cdots, m,$$

$$\zeta^j(z)=\beta^j(x), \qquad j=1, \cdots\cdots, \ell$$

for every $z={}^t(e,\ x)\in R^1\times G$.[2] Then $\xi^i(z)$, $i=1, \cdots\cdots, m$ and $\zeta^j(t)$, $j=1, \cdots\cdots, \ell$ are real valued functions defined on $R^1\times G$ and of class C^1 with respect to $z \in R^1\times G$. Let

$$\varphi(z)=\gamma(x)+x^0, \qquad \text{for every} \qquad z={}^t(x^0,\ x) \in R^1\times G.$$

Let $\mathscr{X}$ denotes the space of af all $n+1$ dimensicnal vector valued continuous functions defined on J with sup. norm topology, *i. e.*,

$$\| z \| = \sup_{t\in J} | z(t) | = \max_{t\in J} | z(t) | .$$

Then the space $\mathscr{X}$ is a real Banach space. Let $\mathscr{Y}_1=R^{m+1}$ and $\mathscr{Y}_2=R^{\ell+1}$, with ordinary Euclidian norm. Let C_1 and C_2 be the subsets defined by

$$C_1=\{e={}^t(e^0,\ e^1,\ \cdots\cdots,\ e^m)\ ;\ \ e^i\leqq 0,\ i=0,\ 1,\ \cdots\cdots,\ m\},$$

$$C_2=\{e={}^t(e^0,\ e^1,\ \cdots\cdots,\ e^\ell)\ ;\ \ e^0\geqq 0,\ e^j\leqq 0,\ j=1,\ \cdots\cdots,\ \ell\},$$

then the ses C_1 and C_2 are closed convex cones in $\mathscr{Y}_1$ and $\mathscr{Y}_2$, such that $\overline{\{C_1{}^0\}}=C_1$ and $\overline{\{C_2{}^0\}}=C_2$.[3]

Now, let us assume that the differential game given in secton 2 has the saddle point $(\bar{u}(t),\ \bar{v}(t))\in\Omega_u\times\Omega_v$, and the $\bar{x}(t)$, $t \in J$, is the trajectry corresponding to $(\bar{u}(t),\ \bar{v}(t))$, *i. e.*, the $\bar{x}(t)$, $t\in J$, is the optimal trajectory, and let $\bar{x}^0(t)$, $t\in J$, be the solution of (3・1) corresponding to $(\bar{u}(t),\ \bar{v}(t))$. Let $\bar{z}(t)={}^t(\bar{x}^0(t),\ \bar{x}(t))$, $t\in J$, then $\bar{z}(t)$, $t\in J$, is a solution of (3・2) corresponding to $(\bar{u}(t),\ \bar{v}(t))$.

Let A_1 and A_2 be the subsets in $\mathscr{X}$ defined by

$$A_1=\left\{ z\in\mathscr{X}\ ;\ \frac{dz(t)}{dt}=F(z(t),\ u(t),\ \bar{v}(t),\ t),\ u\in\Omega_u,\ z(\tau_1)=z_0,\ t\in J\right\},$$

$$A_2=\left\{ z\in\mathscr{X}\ ;\ \frac{dz(t)}{dt}=F(z(t),\ \bar{u}(t),\ v(t),\ t),\ v\in\Omega_v,\ z(\tau_1)=z_0,\ t\in J\right\},$$

respectively, and let G_1 and G_2 be the functions from $\mathscr{X}$ into $\mathscr{Y}_1$ and $\mathscr{Y}_2$ defined by

$$G_1(z)=\begin{pmatrix} \varphi(z(\tau_2))-\varphi(\bar{z}(\tau_2)) \\ \xi^1(z(\tau_2)) \\ \vdots \\ \xi^m(z(\tau_2)) \end{pmatrix},$$

$$G_2(z)=\begin{pmatrix} \varphi(z(\tau_2))-\varphi(\bar{z}(\tau_2)) \\ \zeta^1(z(2)) \\ \vdots \\ \zeta^\ell(z(\tau_2)) \end{pmatrix},$$

respectively. Since $\bar{u}(t)\in\Omega_u$ and $\bar{v}(t)\in\Omega_v$, $\bar{z}\in A_1$ and $\bar{z}\in A_2$, *i. e.*,

$$\bar{z}\in A_1\cap A_2. \qquad (3\cdot 3)$$

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2) The symbol ${}^t(\cdot)$ means the transposed matrix of the matrix $(\cdot)$.

3) The symbol S^0 means the interior of the set S.
The symbol $\bar{S}$ means the closure of the set S.

Since $\bar{u}(t)$, $\bar{v}(t)$ are the optimal strategies of player $P1$ and $P2$, respectively, so

$\xi^i(\bar{z}(\tau_2))\leqq 0$, $i=1$, ………, m,

$\zeta^j(\bar{z}(\tau_2))\leqq 0$, $j=1$, ………, ℓ,

$\varphi(\bar{z}(\tau_2))-\varphi(\bar{z}(\tau_2))=0$.

By virtue of the definitions of G_1, G_2, C_1 and C_2,

$$G_1(\bar{z}) \in C_1 \text{ and } G_2(\bar{z}) \in C_2. \qquad (3\cdot4)$$

Let $z_1 \in A_1$ and $z_1 \in A_2$. By the definitions of optimal strategies,

$\xi^i(z_2(\tau_2))\leqq 0$, $i=1$, …………, m, $\quad \varphi(z_2(\tau_2))-\varphi(\bar{z}(\tau_2))\leqq 0$,

$\zeta^j(z_1(\tau_2))\leqq 0$, $j=1$, ………, ℓ, $\quad \varphi(z_1(\tau_2))-\varphi(\bar{z}(\tau_2))\geqq 0$.

It follows that

$$G_1(z_2)\in C_1 \quad \text{and} \quad G_2(z_1) \in C_2.$$

Therefore,

$$G_1(z_2)\in C_1, \quad \text{for every} \quad z_2 \in A_2, \qquad (3\cdot5)$$

$$G_2(z_1)\in C_2, \quad \text{for every} \quad z_1 \in A_1. \qquad (3\cdot6)$$

Suppose that there exists a $z' \in A_1$ such that $G(z') \in C_1{}^0$, Then

$\xi^i(z'(\tau_2))<0$, $i=1$, ……, m and $\varphi(z'(\tau_2))<\varphi(\bar{z}(\tau_2))$.

Therefore, the $z'(t)={}^t(x'^0(t),\ x'(t))$, $t \in J$, satisfies that

$$\frac{dx'(t)}{dt}=f(x'(t),\ u'(t),\ \bar{v}(t),\ t),\ t\in J,\ u' \in \Omega u,$$

$\alpha^i(x'(\tau_2))<0$, $i=1$, ……, m,

and

$P(u',\ \bar{v})<P(\bar{u},\ \bar{v})$,

where,

$$P(u',\ \bar{v})=\gamma(x'(\tau_2))+\int_{\tau_1}^{\tau_2} g(x'(t),\ u'(t),\ \bar{v}(t),\ t)dt.$$

This contradicts the fact that $\bar{u}(t) \in \Omega_u$ is the optimal strategy of player $P1$. Therefore,

$$\{z\ ;\ G_1(z)\in C_1{}^0,\ z\in A_1\}=\phi.^{4)} \qquad (3\cdot7)$$

Similarly,

$$\{z\ ;\ G_2(z)\in C_2{}^0,\ z\in A_2\}=\phi. \qquad (3\cdot8)$$

Since $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are real Banach spaces, by (3・3) — (3・8), and Ref. 1, Definition 3, the $\bar{z}\in \mathscr{X}$ is the $(G_1,\ G_2,\ A_1,\ A_2)$-saddle point. That is, the following lemma holds.

Lemma. 3・2 *If the differential game introduced in section 2 has the saddle point* $(\bar{u}(t),\ \bar{v}(t))\in\Omega_u\times\Omega_v$, *and if a* $\bar{z}(t)$, $t \in J$ *is the solution of the differential equation* (3・2) *corresponding to* $\bar{u}(t)$ *and* $\bar{v}(t)$, *then* $\bar{z}$ *is the* $(G_1,\ G_2,\ A_1,\ A_2)$*-saddle point introduced in Ref.* 1.

For $z(t)=(z^0(t),\ z^1(t),\ \cdots\cdots,\ z^n(t)) \in \mathscr{X}$, let

$$\varphi_z=\left(\frac{\partial\varphi(z(\tau_2))}{\partial z^0},\ \frac{\partial\varphi(z(\tau_2))}{\partial z^1},\ \cdots\cdots\cdots\cdots,\ \frac{\partial\varphi(z(\tau_2))}{\partial z^n}\right), \qquad (3\cdot9)$$

$$\xi_z^i=\left(\frac{\partial\xi^i(z(\tau_2))}{\partial z^0},\ \frac{\partial\xi^i(z(\tau_2))}{\partial z^1},\ \cdots\cdots\cdots\cdots,\ \frac{\partial\xi^i(z(\tau_2))}{\partial z^n}\right),\ i=1,\ \cdots\cdots,\ m, \qquad (3\cdot10)$$

$$\zeta_z^j=\left(\frac{\partial\zeta^j(z(\tau_2))}{\partial z^0},\ \frac{\partial\zeta^j(z(\tau_2))}{\partial z^1},\ \cdots\cdots\cdots\cdots\cdots,\ \frac{\partial\zeta^j(z(\tau_2))}{\partial z^n}\right),\ j=1,\ \cdots\cdots\cdots,\ \ell, \qquad (3\cdot11)$$

4) The symbol ϕ means the empty set.

and let

$$g_{1_z}=\begin{pmatrix}\varphi_z\\ \xi_z^1\\ \vdots\\ \xi_z^m\end{pmatrix},\qquad g_{2_z}=\begin{pmatrix}\varphi_z\\ \zeta_z^1\\ \vdots\\ \zeta_z^l\end{pmatrix}. \tag{3・12}$$

Then, the following lemma holds.

Lemma. 3・3 *At the point $\bar{z}\in\mathscr{Z}$, the following relations are satisfied;*

$$\frac{G_1(\bar{z}+\varepsilon y)-G_1(\bar{z})}{\varepsilon}\xrightarrow[\substack{\varepsilon\to 0\\ y\to z}]{} g_{1_{\bar{z}}},\ \text{for every } z\in\mathscr{Z},$$

$$\frac{G_2(\bar{z}+\varepsilon y)-G_2(\bar{z})}{\varepsilon}\xrightarrow[\substack{\varepsilon\to 0\\ y\to z}]{} g_{2_{\bar{z}}}\ \text{for every } z\in\mathscr{Z},$$

and the functions $g_{1_{\bar{z}}}$ and $g_{2_{\bar{z}}}$ are linear continuous functons from $\mathscr{Z}$ into $\mathscr{Y}_1$ and $\mathscr{Y}_2$, respectively.

Proof. Since the functions ξ^i, $i=1,\cdots\cdots,m,$ and φ are of class C^1, the following relation is satisfied;

$$G_1(\bar{z}+\varepsilon y)-G_1(\bar{z})=\frac{\partial G_1(\bar{z}(\tau_2))}{\partial z}\varepsilon y+o(\varepsilon y),\quad \text{for every}\quad \varepsilon>0,\ y\in\mathscr{Z},$$

where $o(\varepsilon y)/\varepsilon\longrightarrow 0$ as $\varepsilon\to 0$ uniformly. Since $\varphi(\bar{z}(\tau_2))$ is constant,

$$\frac{\partial G_1(\bar{z}(\tau_2))}{\partial z}=g_{1_{\bar{z}}}.$$

Therefore

$$\frac{G_1(\bar{z}+\varepsilon y)-G_1(\bar{z})}{\varepsilon}=g_{1_{\bar{z}}}(y)+\frac{o(\varepsilon y)}{\varepsilon}\quad \text{for every } \varepsilon>0,\ y\in\mathscr{Z},$$

Then

$$\frac{G_1(\bar{z}+\varepsilon y)-G_1(\bar{z})}{\varepsilon}\xrightarrow[\substack{\varepsilon\to 0\\ y\to z}]{} g_{1_{\bar{z}}}(z),\quad \text{for every } z\in\mathscr{Z}.$$

Similarly,

$$\frac{G_2(\bar{z}+\varepsilon y)-G_2(\bar{z})}{\varepsilon}\xrightarrow[\substack{\varepsilon\to 0\\ y\to z}]{} g_{2_{\bar{z}}}(z),\quad \text{for every } z\in\mathscr{Z}.$$

It is obvious that $g_{1_{\bar{z}}}$ and $g_{2_{\bar{z}}}$ are linear continuous functions from $\mathscr{Z}$ into $\mathscr{Y}_1$ and $\mathscr{Y}_2$, respectively, because $g_{1_{\bar{z}}}$ and $g_{2_{\bar{z}}}$ are $(m+1)\times(n+1)$ and $(\ell+1)\times(n+1)$ matricies, respectively. (Q. E. D.)

Let H_1 and H_2 be the sets defined by

$$H_1=\{F(z,\ u(t),\ \bar{v}(t),\ t)\ ;\ u(t)\in\Omega_u,\ t\in I\},$$

$$H_2=\{F(z,\ \bar{u}(t),\ v(t),\ t)\ ;\ v(t)\in\Omega_v,\ t\in I\}.$$

By Gamkrelize, H_1 and H_2 are quasiconvex (Ref. 2).

Let us consider the following linear variational equation of (3・2) along the solution $\bar{z}(t)$;

$$\frac{d\delta z_1(t)}{dt}=\frac{\partial F(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)}{\partial z}\delta z_1(t)+\delta F_1(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t),\quad t\in I,$$

$$\delta F_1\in[H_1]-\bar{F},\ \delta z_1(\tau_1)=0\in R^{n+1}, \tag{3・13}$$

$$\frac{d\delta z_2(t)}{dt}=\frac{\partial F(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)}{\partial z}\delta z_2(t)+\delta F_2(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t),\quad t\in I,$$

$$\delta F_2\in[H_2]-\bar{F},\ \delta z_2(\tau_1)=0\in R^{n+1}, \tag{3・14}$$

where $[H_1]$ and $[H_2]$ are convex hull of the families H_1 and H_2, respectively, and $\bar{F}=F(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)$. Let $\Phi(t)$ be a nonsingular matrix function that satisfies the equation ;

$$\frac{d\Phi(t)}{dt}=\frac{\partial F(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)}{\partial z}\Phi(t), \qquad \Phi(\tau_1)=E, \tag{3・15}$$

where E is the $(n+1)\times(n+1)$ identity matrix, then the solutions of (3・13) and (3・14) are given by

$$\delta z_1(t)=\Phi(t)\int_{\tau_1}^{t}\Phi^{-1}(s)\delta F_1(\bar{z}(s),\ \bar{u}(s),\ \bar{v}(s),\ s)ds,$$

$$\delta z_2(t)=\Phi(t)\int_{\tau_1}^{t}\Phi^{-1}(s)\delta F_2(\bar{z}(s),\ \bar{u}(s),\ \bar{v}(s),\ s)ds.$$

Let K_1 and K_2 be the subsets in $\mathscr{X}$ defined by

$$K_1=\{\delta z_1\in\mathscr{X}\ ;\ \delta z_1(t)=\Phi(t)\int_{\tau_1}^{t}\Phi^{-1}(s)\delta F_1(\bar{z}(s),\ \bar{u}(s),\ \bar{v}(s),\ s)ds,\ \ \delta F_1\in[H_1]-\bar{F},\ t\in J\},$$

$$K_2=\{\delta z_2\in\mathscr{X}\ ;\ \delta z_2(t)=\Phi(t)\int_{\tau_1}^{t}\Phi^{-1}(s)\delta F_2(\bar{z}(s),\ \bar{u}(s),\ \bar{v}(s),\ s)ds,\ \ \delta F_2\in[H_2]-\bar{F},\ t\in J\}.$$

Then the following lemma holds.

Lemma. 3・4 *The subsets K_1 and K_2 be convex sets such that*

$0\in K_1\subset LC(A_1,\ \bar{z})$ *and* $0\in K_2\subset LC(A_2,\ \bar{z})$,

where $LC(A_i,\ \bar{z})$, $i=1,2$, mean the local cones of A_i, $i=1,2$, at $\bar{z}$ defined by Ref. 1, *Definition* 2.

Proof. Since $\bar{F}\in H_1$, it follows that $0\in[H_1]-\bar{F}$. Therefore, $0\in K_1$. Because of convexity of the set $[H_1]-F$, it is obvious that K_1 is convex set. Let $\delta z_1\in K_1$, *i. e.*,

$$\delta z_1(t)=\Phi(t)\int_{\tau_1}^{t}\Phi^{-1}(s)\delta F_1(\bar{z}(s),\ \bar{u}(s),\ \bar{v}(s),\ s)ds,\quad t\in J,$$

where $\delta F_1(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)\in[H_1]-\bar{F}$. Since H_1 is quasiconvex, there exists, for every $\varepsilon\in(0,\ 1]$, a function $k_\varepsilon(z,\ t)$ from $R^1\times G\times I$ to R^{n+1}, in class C^1 with respect to z, and depending on δF_1 and ε, such that

$$(\bar{F}+\varepsilon\delta F_1+k_\varepsilon)\in H_1 \tag{3・16}$$

(see Ref. 2, pp. 111). Now let us consider the perturbed equation

$$\frac{dz(t)}{dt}=F(z(t),\ \bar{u}(t),\ \bar{v}(t),\ t)+\varepsilon\delta F_1(z(t),\ \bar{u}(t),\ \bar{v}(t),\ t)+k_\varepsilon(z(t),\ t). \tag{3・17}$$

It is not difficult to show that, if $\varepsilon>0$ is sufficiently small, then the solution $z(t)$ of (3・17), satisfying the initial condition $z(\tau_1)=\bar{z}(\tau_1)$, exists for $\tau_1\leqq t\leqq\tau_2$, and has the form

$$z(t)=\bar{z}(t)+\varepsilon\delta z_1(t)+o(\varepsilon),\qquad \tau_1\leqq t\leqq\tau_2. \tag{3・18}$$

where $o(\varepsilon)/\varepsilon\longrightarrow 0$ as $\varepsilon\longrightarrow 0$ uniformly in t, $\tau_1\leqq t\leqq\tau_2$. By virtue of (3・16), $z(t)\in A_1$. Therefore,

$$z\longrightarrow\bar{z},\quad \frac{1}{\varepsilon}(z-\bar{z})\longrightarrow\delta z_1 \qquad \text{as} \qquad \varepsilon\longrightarrow 0.$$

By Varaiya (Ref. 3),

$\delta z_1 \in LC(A_1, \bar{z})$.

Therefore, K_1 is convex set such that

$0 \in K_1 \subset LC(A_1, \bar{z})$.

Similarly, we can show that K_2 is convex set such that

$0 \in K_2 \subset LC(A_2, \bar{z})$. (Q. E. D.)

By the Lemma. 3・2, 3・3, 3・4, and Ref. 1, Theorem, the following lemma holds.

Lemma. 3・5. *There exist $y_1^* \in \mathscr{Y}_1^*$ and $y_2^* \in \mathscr{Y}_2^*$, not both zero, such that*[5)]

$$y_1^*(g_{1\bar{z}}(z_1)) + y_2^*(g_{2\bar{z}}(z_1)) \geqq 0 \quad \text{for every} \quad z_1 \in K_1, \tag{3・19}$$

$$y_1^*(g_{1\bar{z}}(z_2)) + y_2^*(g_{2\bar{z}}(z_2)) \leqq 0 \quad \text{for every} \quad z_2 \in K_2, \tag{3・20}$$

$$y_1^*(G_1(\bar{z})) = 0, \tag{3・21}$$

$$y_2^*(G_2(\bar{z})) = 0, \tag{3・22}$$

$$y_1^*(y_1) \leqq 0 \quad \text{for every} \quad y_1 \in C_1, \tag{3・23}$$

$$y_2^*(y_2) \geqq 0 \quad \text{for every} \quad y_2 \in C_2. \tag{3・24}$$

4. Necessary conditions for the Saddle Point. In this section, we shall formulate the the nacessary conditions and give its proof by use of the results obtained in section 3.

Theorem. *Let $(\bar{u}(t), \bar{v}(t)) \in \Omega_u \times \Omega_v$ be the saddle point of the differential game formulated in section 2, and let $\bar{z}(t)$, $\tau_1 \leqq t \leqq \tau_2$, be the solution of (3・2) corresponding to $\bar{u}(t)$, $\bar{v}(t)$. Then there exists an absolutely continuous vector valued function $\psi(t)$, $\tau_1 \leqq t \leqq \tau_2$, and Hamiltionian functian $H(\psi, z, u, v, t)$, such that $\bar{z}(t)$, $\psi(t)$, $\tau_1 \leqq t \leqq \tau_2$, satisfy the following Hamiltonian system of equations for almost all t, $\tau_1 \leqq t \leqq \tau_2$:*

$$\frac{d\bar{z}(t)}{dt} = \frac{\partial H(\psi(t), \bar{z}(t), \bar{u}(t), \bar{v}(t), t)}{\partial \psi} = F(\bar{z}(t), \bar{u}(t), \bar{v}(t), t), \tag{4・1}$$

$$\frac{d\psi(t)}{dt} = -\frac{\partial H(\psi(t), \bar{z}(t), \bar{u}(t), \bar{v}(t), t)}{\partial z} = -\psi(t)\frac{\partial F(\bar{z}(t), \bar{u}(t), \bar{v}(t), t)}{\partial z}, \tag{4・2}$$

and such that the inequalities

$$H(\psi(t), \bar{z}(t), \bar{u}(t), v, t) \leqq H(\psi(t), \bar{z}(t), \bar{u}(t), \bar{v}(t), t) \leqq H(\psi(t), \bar{z}(t), u, \bar{v}(t), t), \tag{4・3}$$

hold for every $u \in U$ and $v \in V$. Further, $\psi(t)$, $\tau_1 \leqq t \leqq \tau_2$, satisfies the terminal condition:

$$\psi(\tau_2) = \rho\varphi_{\bar{z}} + \sum_{i=1}^{m} \nu_i \xi^i_{\bar{z}} + \sum_{j=1}^{\ell} \mu^j \zeta^j_{\bar{z}}, \tag{4・4}$$

where ν^i, $i=1, \cdots\cdots, m$, μ^j, $j=1, \cdots\cdots, \ell$ and ρ are real numbers and $\xi^i_{\bar{z}}$, $i=1, \cdots\cdots, m$, $\zeta^j_{\bar{z}}$, $j=1, \cdots\cdots$. ℓ and $\varphi_{\bar{z}}$ are partial derivatives of ξ^i, $i=1, \cdots\cdots, m$, ζ^j, $j=1, \cdots\cdots \ell$ and φ at the point $\bar{z}(\tau_2)$.

Proof. Since spaces $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are R^{m+1} and $R^{\ell+1}$, respectively, $\mathscr{Y}_1 = \mathscr{Y}_1^* = R^{m+1}$ and $\mathscr{Y}_2 = \mathscr{Y}_2^* = R^{\ell+1}$. Therefore, by (3・12), (3・19) and (3・20),

$$(\nu^0 + \mu^0)\varphi_{\bar{z}}(z_1) + \sum_{i=1}^{m} \nu^i \xi^i_{\bar{z}}(z_1) + \sum_{j=1}^{\ell} \mu^j \zeta^j_{\bar{z}}(z_1) \geqq 0 \quad \text{for every} \quad z_1 \in K_1, \tag{4・5}$$

$$(\nu^0 + \mu^0)\varphi_{\bar{z}}(z_2) + \sum_{i=1}^{m} \nu^i \xi^i_{\bar{z}}(z_2) + \sum_{j=1}^{\ell} \mu^j \zeta^j_{\bar{z}}(z_2) \leqq 0 \quad \text{for every} \quad z_2 \in K_2, \tag{4・6}$$

5) The symbol $\mathscr{Y}^*$ represents the conjugate space of $\mathscr{Y}$.

where $y_1^* = {}^t(\nu^0, \nu^1, \cdots\cdots, \nu^m) \in R^{m+1}$, $y_2^* = {}^t(\mu^0, \mu^1, \cdots\cdots, \mu^l) \in R^{l+1}$.
By the definitions K_1 and K_2,

$$\pi\Phi(\tau_2)\int_{\tau_1}^{\tau_2}\Phi^{-1}(s)\delta F_1(\bar{z}(s),\ \bar{u}(s),\ \bar{v}(s),\ s)ds \geqq 0 \qquad (4\cdot 7)$$

for every $\delta F_1 \in [H_1] - \bar{F}$, and

$$\pi\Phi(\tau_2)\int_{\tau_1}^{\tau_2}\Phi^{-1}(s)\delta F_2(\bar{z}(s),\ \bar{u}(s),\ \bar{v}(s),\ s)ds \leqq 0 \qquad (4\cdot 8)$$

for every $\delta F_2 \in [H_2] - \bar{F}$, where π is the $n+1$ dimensional vector:

$$\pi = (\nu^0 + \mu^0)\varphi_{\bar{z}} + \sum_{i=1}^{m} \nu^i \xi^i_{\bar{z}} + \sum_{j=1}^{l} \mu^j \zeta^j_{\bar{z}}\ .$$

Now let us defined the function ψ(t) as follows:

$$\psi(t) = \pi\Phi(\tau_2)\Phi^{-1}(t), \qquad t \in J. \qquad (4\cdot 9)$$

By (4・7), (4・8), (3・15) and (4・9), the function $\psi(t)$ satisfies the following relations:

$$\int_{\tau_1}^{\tau_2}\psi(t)\delta F_1(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)dt \geqq 0 \qquad (4\cdot 10)$$

for every $\delta F_1 \in [H_1] - \bar{F}$,

$$\int_{\tau_1}^{\tau_2}\psi(t)\delta F_2(\bar{z}(t),\ \bar{u}(t).\ \bar{v}(t),\ t)dt \leqq 0 \qquad (4\cdot 11)$$

for every $\delta F_2 \in [H_2] - \bar{F}$,

$$\frac{d\psi(t)}{dt} = -\psi(t)\frac{\partial F(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)}{\partial z} \qquad \text{for almost all } t \in J, \qquad (4\cdot 12)$$

$$\psi(\tau_2) = \pi. \qquad (4\cdot 13)$$

Now, let $H(\psi(t),\ z(t),\ \bar{u}(t),\ \bar{v}(t),\ t)$, $t \in J$, be the Hamiltonian function defined by

$$H(\psi(t),\ z(t),\ \bar{u}(t),\ \bar{v}(t),\ t) = \psi(t)F(z(t),\ \bar{u}(t),\ \bar{v}(t),\ t). \qquad (4\cdot 14)$$

Since $\bar{z}(t)$, $t \in J$, is the solution of (3・2) corresponding to $\bar{u}(t)$ and $\bar{v}(t)$, and $\psi(t)$, $t \in J$, is the solution of (4・12), the relations (4・1) and (4・2) are satisfied on J. Let ρ be the real number such that $\rho = \nu^0 + \mu^0$, then $\psi(t)$ satisfies the relation (4・4) at $t = \tau_2$.
From (4・10), (4・11) and from the definitions of H_1 and H_2,

$$\int_{\tau_1}^{\tau_2}\psi(t)F(\bar{z}(t),\ u(t),\ \bar{v}(t),\ t)dt \geqq \int_{\tau_2}^{\tau_1}\psi(t)F(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)dt, \qquad (4\cdot 15)$$

$$\int_{\tau_1}^{\tau_2}\psi(t)F(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t)dt \geqq \int_{\tau_2}^{\tau_1}\psi(t)F(\bar{z}(t),\ \bar{u}(t),\ v(t),\ t)dt, \qquad (4\cdot 16)$$

for every $u(t) \in \Omega_u$ and $v(t) \in \Omega_v$. The function $F(z,\ u,\ v,\ t)$ is continuous with respect to u, v, and t. By Gamkrelize, the relations (4・15) and (4・16) mean that

$$\psi(t)F(\bar{z}(t),\ u,\ \bar{v}(t),\ t) \geqq \psi(t)F(\bar{z}(t),\ \bar{u}(t),\ \bar{v}(t),\ t) \geqq \psi(t)F(\bar{z}(t),\ \bar{u}(t),\ v,\ t),$$

for almost all $t \in J$. these follow that the relations (4・3) hold. This completes the proof of the theorem.

(Q. E. D.)

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