A GENERAL THEORY OF SADDLE POINT

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Abstract

This article is devoted to the formulation of a general saddle point of a game and to the derivation of necessary conditions which the saddle point must satisfy. The necessary conditions include the theory of various games and differential games.

1. Introduction. The study of game theory has been initiated by John von Neumann and Osker Morgenstern (Ref. 1). Then, the game theory has been developed by mathematicians and economists. In 1954, Isaacs applied the game theory to differential systems, and this is the differential game (Ref. 2). After that, useful works in the branch of the differential game have been achieved by L. S. Pontryagin, L. D. Berkovitz and others (Ref. 3 − 6).

This paper is devoted to a general theory of games which includes a variety of theory of games and differential games. That is, in the case that a game or a differential game has saddle point, a (G_1, G_2, A_1, A_2) -saddle point which is abstraction of the saddle point in a game and a differential game is defined, and its necessary conditions is presented. The necessary conditions will be applied to various games and differential games.

In section 2, some definitions of terminology and preliminary results are present. In section 3, the abstract saddle point is defined, and the necessary conditions which is satisfied by the saddle point is proved.

2. Preliminary Results. In this section, we shall introduce some definitions and derive lemmas which will be used for the proof of the main theorem in the following section.

Let $\mathscr{X}$ and $\mathscr{Y}$ be real Banach spaces, let A be a nonempty subset of $\mathscr{X}$, and let $\bar{x} \in A$.

Definition 1. By the closed cone of A at $\bar{x}$, we mean the intersection of all closed cone [1] containing the set

$$A-\bar{x} = \{ a-\bar{x} : a \in A \}.$$

We denote this set by $C(A, \bar{x})$.

Definition 2. By the local closed cone of A at $\bar{x}$, we mean the set

$$LC(A, \bar{x}) = \bigcap_{N \in U(\bar{x})} C(A \cap N, \bar{x}),$$

where $U(\bar{x})$ is the class of all neiborhoods of $\bar{x}$.

By Varaiya, the following lemma holds (Ref. 7).

Lemma 1. Let $z \neq 0$. The element $z \in LC(A, \bar{x})$ if, and only if, there exist $\{ x_n : x_n \in A, n = 1, 2, \cdots\cdots\}$ and $\{ \lambda_n : \lambda_n > 0, n = 1, 2, \cdots\cdots\}$, such that

$$x_n \longrightarrow \bar{x}, \ \lambda_n(x_n - \bar{x}) \longrightarrow z \text{ as } n \longrightarrow \infty.$$

Let C be a nonempty closed convex cone in $\mathscr{Y}$ such that $\overline{\{C^0\}} = C$ [2]. Then the following lemma

1 A set C is a cone if, and only if, $\sigma C \subset C$ for all $\sigma \geqq 0$.

2 The symbol S^0 means the interior of the set S.

The symbol $\bar{S}$ means the closure of the set S.

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holds (Ref. 8).

Lemma 2. If $y_1 \in C^0$ and $y_2 \in C$, then $y_1 + y_2 \in C^0$.

Now, let us consider a mapping G from $\mathscr{X}$ into $\mathscr{Y}$. We assume that G is continuous and that, for every $z \in \mathscr{X}$, there exists a linear continuous mapping g_z from $\mathscr{X}$ into $\mathscr{Y}$ such that

$$\frac{G(z+\varepsilon y)-G(z)}{\varepsilon} \xrightarrow[\substack{\varepsilon \to 0 \\ y \to x}]{} g_z(x) \text{ for every } x \in \mathscr{X}. \tag{1}$$

Lemma 3. Let $\bar{x} \in A$ and $G(\bar{x}) \in C$. Then

$\{x : G(x) \in C^0, \ x \in A\} = \phi$ [3] implies $LC(A, \bar{x}) \cap g_{\bar{x}}^{-1}(C^0 - G(\bar{x})) = \phi$.

Proof. Let $x \in LC(A, \bar{x}) \cap g_{\bar{x}}^{-1}(C^0 - G(\bar{x}))$. If $x = 0$, $g_{\bar{x}}(0) \in C^0 - G(\bar{x})$, that is, $G(\bar{x}) \in C^0$. Since $\bar{x} \in A$, $\bar{x} \in \{x : G(x) \in C^0, \ x \in A\}$.

Therefore, if $x = 0$, this lemma folds.

We assume that $x \neq 0$. Since $x \in LC(A, \bar{x})$, by Lemma 1, there exist $\{x_n : x_n \in A, \ n = 1, 2, \cdots\cdots\}$ and $\{\lambda_n : \lambda_n > 0, \ n = 1, 2, \cdots\cdots\}$ such that

$$x_n \longrightarrow \bar{x}, \ \lambda_n(x_n - \bar{x}) \longrightarrow x \quad \text{as} \quad n \longrightarrow \infty.$$

Since $1/\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$, it follows from (1) that

$$\frac{G\left(\bar{x} + \frac{\lambda_n(x_n - \bar{x})}{\lambda_n}\right) - G(\bar{x})}{1/\lambda_n} \xrightarrow[n \to \infty]{} g_{\bar{x}}(x),$$

that is,

$$\lambda_n(G(x_n) - G(\bar{x})) \xrightarrow[n \to \infty]{} g_{\bar{x}}(x). \tag{2}$$

On the other hand, $x \in g_{\bar{x}}^{-1}(C^0 - G(\bar{x}))$. It follows that

$$g_{\bar{x}}(x) \in C^0 - G(\bar{x}). \tag{3}$$

Since $C^0 - G(\bar{x})$ is open set, from (2) and (3), there exists a positive number N such that

$$n > N \text{ implies } \lambda_n(G(x_n) - G(\bar{x})) \in C^0 - G(\bar{x}),$$

that is,

$$n > N \text{ implies } \lambda_n(G(x_n) - (\lambda_n - 1)G(\bar{x})) \in C^0. \tag{4}$$

There is a positive number M such that $n > M$ implies $\lambda_n > 1$, because $\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Since C is a cone and $G(\bar{x}) \in C$,

$$n > M \text{ implies } (\lambda_n - 1)G(\bar{x}) \in C. \tag{5}$$

Therefore, from (4), (5) and Lemma 2, it follows that

$$n > max\{N, M\} \quad \text{implies} \quad G(x_n) \in C^0.$$

Since $x_n \in A$,

$$n > max\{N, M\} \text{ implies } x_n \in \{x : G(x) \in C^0, \ x \in A\},$$

and this contradicts the fact $\{x : G(x) \in C^0, \ x \in A\} = \phi$. (Q. E. D.)

3. The Concept of Abstract Saddle Point and Game Theory. In this section, We shall introduce the concept of abstract saddle point, and present necessary conditions which the saddle point must satisfy.

In this section, let $\mathscr{Y}_1$, $\mathscr{Y}_2$ and $\mathscr{X}$ be real Banach spaces. In order to define (G_1, G_2, A_1, A_2)-saddle point, there must given:

3 The symbol ϕ denote the empty set.

(i) closed convex cones $C_1 \subset \mathscr{Y}_1$ and $C_2 \subset \mathscr{Y}_2$ such that $\overline{\{C_1^0\}} = C_1$ $\overline{\{C_2^0\}} = C_2$,

(ii) arbitrary subsets A_1 and A_2 in $\mathscr{X}$,

(iii) continuous funetions G_1 from $\mathscr{X}$ into $\mathscr{Y}_1$ and G_2 from $\mathscr{X}$ into $\mathscr{Y}_2$

Definition 3. $\hat{x} \in \mathscr{X}$ be called a (G_1, G_2, A_1, A_2) -saddle point if

① $\hat{x} \in A_1 \cap A_2$,

② $G_1(\hat{x}) \in C_1$ and $G_2(\hat{x}) \in C_2$,

③ $G_1(x_2) \in C_1$ for every $x_2 \in A_2$ and $G_2(x_1) \in C_2$ for every $x_1 \in A_1$,

④ $\{x ; G_1(x) \in C_1^0,\ x \in A_1\} = \phi$ and $\{x : G_2(x) \in C_2^0,\ x \in A_2\} = \phi$.

In order to obtain a meaningful necessary conditions for the saddle point, the following is satisfied by the functions G_1, G_2, the subsets A_1, A_2, cones C_1, C_2, and the (G_1, G_2, A_1, A_2)-saddle point.

Condition. At the point $\hat{x} \in \mathscr{X}$, there exist linear continuous functions $g_{1\hat{x}}$ from $\mathscr{X}$ into $\mathscr{Y}_1$ and $g_{2\hat{x}}$ from $\mathscr{X}$ into $\mathscr{Y}_2$ such that

$$\frac{G_1(\hat{x}+\varepsilon y) - G_1(\hat{x})}{\varepsilon} \xrightarrow[\substack{\varepsilon \to 0 \\ y \to x}]{} g_{1\hat{x}}(x) \text{ for every } x \in \mathscr{X}, \tag{6}$$

$$\frac{G_2(\hat{x}+\varepsilon y) - G_2(\hat{x})}{\varepsilon} \xrightarrow[\substack{\varepsilon \to 0 \\ y \to x}]{} g_{2\hat{x}}(x) \text{ for every } x \in \mathscr{X}. \tag{7}$$

Now, we shall present necessary conditions which the (G_1, G_2, A_1, A_2)-saddle point must satisfy, and give the proof.

Theorem. Let $\hat{x} \in \mathscr{X}$ be a (G_1, G_2, A_1, A_2)-saddle point such that at the point $\hat{x}$, the Condition given in this section is satisfied, and let K_1 and K_2 be arbitrary convex sets such that

$0 \in K_1 \subset LC(A_1, \hat{x})$ and $0 \in K_2 \subset LC(A_2, \hat{x})$.

Then there exist $y_1^* \in \mathscr{Y}_1^*$ and $y_2^* \in \mathscr{Y}_2^*$, not bothzero, such that [4]

$$y_1^*(g_{1\hat{x}}(x_1)) + y_2^*(g_{2\hat{x}}(x_1)) \geqq 0 \quad \text{for every } x_1 \in K_1, \tag{8}$$

$$y_1^*(g_{1\hat{x}}(x_2)) + y_2^*(g_{2\hat{x}}(x_2)) \leqq 0 \quad \text{for every } x_2 \in K_2, \tag{9}$$

$$y_1^*(G_1(\hat{x})) = 0, \tag{10}$$

$$y_2^*(G_2(\hat{x})) = 0, \tag{11}$$

$$y_1^*(y_1) \leqq 0 \quad \text{for every } y_1 \in C_1, \tag{12}$$

$$y_2^*(y_2) \geqq 0 \quad \text{for every } y_2 \in C_2. \tag{13}$$

Proof. For each $x_1 \in K_1$, define a subset $B_1(x_1)$ of $\mathscr{Y}_1$ by

$$B_1(x_1) = \{y : g_{1\hat{x}}(x_1) + G_1(\hat{x}) - y \in C_1\},$$

let

$$B_1 = \bigcup_{x_1 \in K_1} B_1(x_1).$$

Since K_1 is convex, B_1 is a convex subset of $\mathscr{Y}_1$. Also, since $0 \in K_1$ and $G_1(\hat{x}) \in C_1$, it is obvious that $0 \in B_1$ Suppose that $B_1 \cap C_1^0 \neq \phi$, that is, there is an element $y_0 \in B_1 \cap C_1^0$, then there exists a $z_0 \in K_1$ such that

4 The symbol $\mathscr{Y}^*$ represents the conjugete space of $\mathscr{Y}$

$$g_{1\hat{x}}(z_0)+G(\hat{x})-y_0 \in C_1.$$

This means that

$$z_0 \in g_{1\hat{x}}^{-1}\ (C_1^0 - G_1(\hat{x})).$$

On the other hand, $z_0 \in K_1 \subset LC(A_1, x)$. Therefore, by Lemma 3, it follows that

$$\{x : G_1(x) \in C^0,\ x \in A_1\} \neq \phi,$$

and this contradicts that $\hat{x}$ is (G_1, G_2, A_1, A_2)-saddle point. Hence

$$B_1 \cap C^0 = \phi. \tag{14}$$

It is obvious from (14) that $C(B_1, 0) \neq y_1$. Since B_1 is convex, $C(B_1, 0)$ is also convex. Therefore, there exists a nonzero, continuous linear functional $y_1^* \in \mathscr{Y}_1^*$ tangent to $C(B_1, 0)$ at origine, (Ref. 9, p. 425) namely,

$$y_1^*(y) \geqq 0 \qquad \text{for every } y \in B_1. \tag{15}$$

From the definition of B_1, it is obvious that $y \in B_1$ for every $y \in -C_1$. It follows from (15) that

$$y_1^*(y) \leqq 0 \qquad \text{for every } y \in C_1. \tag{16}$$

It is obvious that $G_1(\hat{x}) \in B_1$, so, from (15) we obtain that $y_1(G_1(\hat{x})) \geqq 0$, while, $y_1^*(G_1(\hat{x})) \leqq 0$ because $G_1(\hat{x}) \in C_1$. Thus we obtain that

$$y_1^*(G_1(\hat{x})) = 0. \tag{17}$$

Since $g_{1\hat{x}}(x_1) + G_1(\hat{x}) \in B_1$ for every $x_1 \in K_1$, it follows from (15) that

$$y_1^*(g_{1\hat{x}}(x_1)+G_1(\hat{x})) \geqq 0 \qquad \text{for every } x_1 \in K_1 \tag{18}$$

From (17) and (18), it follows that

$$y_1^*(g_{1\hat{x}}(x_1)) \geqq 0 \qquad \text{for every } x_1 \in K_1. \tag{19}$$

Similary, if we define the set B_2 by

$$B_2 = \bigcup_{x_2 \in K_2} B_2(x_2),$$

where

$$B_2(x_2) = \{y : g_{2\hat{x}}(x_2)+G_2(\hat{x})-y \in C_2,\ x_2 \in K_2\},$$

we obtain that

$$y_2^*(y) \leqq 0 \qquad \text{for every} \qquad y \in B_2,$$

that is,

$$y_2^*(y) \geqq 0 \qquad \text{for every} \qquad y \in C_2, \tag{20}$$

$$y_2^*(G_2(\hat{x})) = 0, \tag{21}$$

$$y_2^*(g_{2\hat{x}}(x_2)) \leqq 0 \quad \text{for every} \qquad x_2 \in K_2. \tag{22}$$

Let $x_2 \in K_2$ and $x_2 \neq 0$. Since $K_2 \subset LC(A_2, \hat{x})$, by Lemma 1, there are $\{x_n : x_n \in A_2,\ n = 1, 2, \cdots\cdots\cdots\}$ and $\{\lambda_n : \lambda_n > 0,\ n = 1, 2, \cdots\cdots\cdots\}$ such that

$$x_n \longrightarrow \hat{x},\ \lambda_n(x_n - \hat{x}) \longrightarrow x_2 \qquad \text{as} \qquad n \longrightarrow \infty.$$

By Definiton 3, ③ and (16), it follows that

$$y_1^*(G_1(x_n) \leqq 0, \qquad n = 1, 2, \cdots\cdots\cdots.$$

From (17) and $x_n = \hat{x} + (1/\lambda_n)(\lambda_n(x_n - \hat{x}))$, $n = 1, 2, \cdots\cdots\cdots$,

$$\frac{y_1^*\left(G_1\left(\hat{x}+\frac{1}{\lambda_n}(\lambda_n(x_n-\hat{x}))\right)\right)-y_1^*\left(G_1(\hat{x})\right)}{\lambda_n}\leqq 0, \quad n=1,\ 2,\ \cdots\cdots\cdots.$$

From (6), this means that

$$y_1^*\left(g_{1\hat{x}}(x_2)\right)\leqq 0.$$

From the fact that $y_1^*\left(g_{1\hat{x}}(0)\right)=0$,

$$x_2\in K_2 \quad \text{implies} \quad y_1^*\left(g_{1\hat{x}}(x_2)\right)\leqq 0. \tag{23}$$

Similary,

$$x_1\in K_1 \quad \text{implies} \quad y_2^*\left(g_{2\hat{x}}(x_1)\right)\geqq 0. \tag{24}$$

From (19) and (24), it follows that

$$y_1^*\left(g_{1\hat{x}}(x_1)\right)+y_2^*\left(g_{2\hat{x}}(x_1)\right)\geqq 0 \quad \text{for every } x_1\in K_1, \tag{25}$$

and from (22) and (23), it follows that

$$y_1^*\left(g_{1\hat{x}}(x_2)\right)+y_2^*\left(g_{2\hat{x}}(x_2)\right)\leqq 0 \quad \text{for every } x_2\in K_2. \tag{26}$$

From(25), (26), (17), (21), (16)and (20), this theorem holds. (Q. E. D.)

4. Concluding Remarks. In the case that a game or a differential game has the saddle point, the theorem shown in Section 3 give the necessary conditions which is satisfied by the saddle point. The theorem is applied to not only a game in a finite demensional space or a multistage game, but also a game in a function space or differential game, because the theorem is discussed in a Banach space.

In case of applying this theorem to a game, the subsets A_1, and A_2 are sets of strategies of each player, and the G_1 and G_2 are a payoff function and constraint conditions of the game. In case of applying it to a multistage game or differential game, the subsets A_1 and A_2 are subsets retaled to strategies of each player, and the G_1 and G_2 are terminal conditions and constraint conditions of the game.

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