ON PRE-ARITHMETICAL LATTICE-ORDERED COMMUTATIVE SEMIGROUPS

KENTARO MURATA

0. Introduction. The aim of the present short paper is to introduce a new algebraic system, named pre-arithmetical lattice-ordered commutative semigroup (abbr. *pal*-semigroup), in which the arithmetic in the sense of Artin can be discussed, and to determine the relationship between the system and the lattice-theoretical divisors investigated in [8].

1. Pal-Semigroups. Let $S = (S, \cdot, \leq)$ be a conditionally complete (upper and lower) lattice-odered semigroup with unity quantity e, $G = (G, \cdot)$ the unit group of $(S, \cdot)$, I the integral part (cone) of $(S, \leq)$ and H the intersection of G and I.

Definition 1. S is called a *pre-arithmetical lattice-ordered semigroup (pal-semigroup)*, if it satisfies the following conditions:

(a) $(S, \cdot)$ *is a quotient semi-group of* I *by* H, *i. e., each element of* S *is expressed as* cx^{-1} *with* c *in* I *and* x *in* H.

(b) H *is a join generator system of* I, *i. e., each element* a *of* I *is written as* $a = sup(E)$, *where* E *is a finite or infinite subset of* H.

For each element a of S we put

$$U(a) = \{u \in G\,;\ u \geq a\}, \qquad L(a) = \{u \in G\,;\ u \leq a\}.$$

Then these two subsets of G are non-void, which are assured by the conditions (a) and (b) respectively. Now it can be shown that for each element a of S there exists a subset A of G such that $a = \sup(A)$, so that we have

$$a = \sup L(a).$$

But $a = \inf U(a)$ is not true in general. Here we consider a map

$$d:\ \ S \longrightarrow S;\quad a \longmapsto d(a) = \inf U(a).$$

Then we have (1) $a \leq d(a)$, (2) $a \leq d(b)$ implies $d(a) \leq d(b)$, (3) $d(d(a)) = d(a)$, and (4) if A is a non-void bounded (lower) subset of G, then $d(\inf(A)) = \inf(A)$.

Definition 2. An element a of S is said to be *divisorial* if $d(a) = a$.

We put $G(a/b) = \{u \in G\,;\, ub \leq a\}$, and $S(a/b) = \{c \in S\,;\, cb \leq a\}$. Then it can be proved that

$$\sup G(a/b) = \sup S(a/b).$$

We denote it by a/b for simplicity. Thus we can see easily that if a is divisorial, then so is a/b for an arbitrary element b of S. Moreover we obtain $d(a) = e/(e/a)$ for every a in S, and

$$d(ab) = d(d(a)d(b)).$$

In the following $d(T)$ will denote the set $\{d(t)\,;\, t \in T\}$ for a non-void subset T of S.

Theorem 1. *The set $d(S)$ of all divisorial elements in S forms a pal-semigroup under the binary operation "$\circ$" defined by $d(a) \circ d(b) = d(d(a)\,d(b))$ and the same ordering in S.*

Proof. It is easy to show that $(d(S), \leq)$ forms a lattice under $d(a) \mathbin{\mathring{\vee}} d(b) = d(d(a) \vee d(b))$ and $d(a) \mathbin{\mathring{\wedge}} d(b) = d(a) \wedge d(b) = d(d(a) \wedge d(b))$, where $\vee$ and $\wedge$ are the join and the meet in $(S, \leq)$ respectively.
If $d(T)$ is a bounded subset (upper) of $d(S)$, then we have that $d(\sup d(T))$ exists, and $d(a) \circ d(\sup d(T)) = d(d(a)\ d(\sup d(T))) = d(a \bullet \sup T) = d(\sup(aT)) = d(\sup(d(a)\ d(T)) = d(\sup(d(a) \circ d(T)))$. Hence "$\circ$" distributes for the join operation $\circ$-sup, which is defined by $\circ\text{-}\sup d(T) := d(\sup d(T))$. Evidently, if $d(T)$ has a lower bound, then $d(T) = d(\inf d(T))$ exists, and it is readily verified that the unit group of $(d(S), \circ)$ contains $(G, \bullet)$ and $(d(S), \circ)$ forms a quotient semigroup of $d(I)$ by $d(H) = H$. Moreover if $a = d(a)$ is in $d(I)$, then we obtain $a = d(\sup E)) = \circ\text{-}\sup(d(E))$ for a subset E of H.

2. Artinian Lattice-Ordered Semigroups.

Definition 3. A pal-semigroup $S=(S, \bullet, \leq)$ is said to be *Artinian*, if and only if $(d(S), \circ)$ forms a group.

Then if S is Artinian, $(d(S), \circ, \leq)$ is a lattice-ordered group and the lattice $d(S)$ satisfies the strong distributive law. Let S be a pal-semigroup.

Definition 4 [8]. The unity quantity e is said to be *integrally closed*, if whenever $u^n \leq c$ for u in G and for all n in N, the positive integers, then $u \leq e$.

This definition is a generalization of the completely integral closedness defined in [2], [3] and [8].

Theorem 2. *A pal-semigroup is Artinian if and only if its unity quantity is intagrally closed.*

Proof. Let S be an Artinian lattice ordered-semigroup. If $\{u^n ; n \in N\}$ has an upper bound, there exists an element x in G such that $u^n \leq x$ for all n in N. Evidently $\{u^{-n} ; n \in N\}$ is bounded (lower) —— x^{-1} is a lower bound ——. Hence there is $\inf_n\{u^n\} =: c$. Since $\{u^n\}$ is contained in G we have $d(c)=c$ by the above property (4) of "d", and we have

$$c \circ u^{-1} = d((\inf_{n\bullet 1} u^{-n})u^{-1} = d(\inf_{n\geq 2} u^{-n}) \geq d(\inf_{n\geq 1} u^{-n}) = c,$$
$$u^{-1} = c^{-1} \circ c \circ u^{-1} \geq c^{-1} \circ c = e.$$

Therefore we have $e \geq u$. Conversely, suppose that e is integrally closed. Let a be an arbitrary element of $d(I)$, and u an element of G such that $u \bullet a(e/a) \leq e$. Then we have $ua \leq e/(e/a) = d(a) = a$, $a \leq u^{-n}a$ for all positive integer n. Hence we get $y \leq u^{-n}a$, $u^n \leq ay^{-1}$ for any y in $L(a)$. This implies $u \leq e$, $e \leq u^{-1}$, and implies $e \leq U(a(e/a))$. On the other hand, since $a(e/a) \leq e$, we have $e \in U(a(e/a))$. This yields the following equalities :

$$a \circ (e/a) = d(a(e/a)) = \inf U(a(e/a)) = e.$$

For any $d(a)$, there exists an element z in H such that $a \circ z \leq e$. By the above equality $a \circ z$ is invertible, so we proved that a is invertible. Q. E. D.

Definition 5. An element a of S is said to be *finitely attainable*, if whenever $a = \sup\ (E)$ for a subset E of G, then there exists a finite number of elements $x_1, \cdots, x_n$ in E such that $a = x_1 \vee \cdots \vee x_n$.

Definition 6. An element a is said to be *d-accessible*, if whenever d $(\sup\ (C)) = a$ for the ascending chain $C = \{c_i \in d(I)\ ;\ c_1 \leq c_2 \leq \cdots\}$, then $\sup C = a$.

Theorem 3. *Let S be an Artinian semigroup. Then a. c. c. (ascending chain condition) holds for the elements in d(I), if and only if unity quantity of S is finitely attainable, and at the same time it is d-accessible.*

Proof. We note that $\sup X = \sup\ (\cup\ \{L(a)\ ;\ a \in X\})$ is valid for an arbitrary non-void subset X of I, where $\cup$ denotes set-theoretic union. Now suppose a. c. c. holds for elements in $d(I)$. If $d(\sup(C)) = e$ for an ascending chain C in $d(I)$, there exists an element c_n in C such that $c_n = e$. This implies $e \leq \sup\ (C) \leq e$, and $\sup\ (C) = e$. It is clear that e is finitely attainable. Conversely, let $a_1 \leq a_2 \leq \cdots$ be any ascending chain in $d(I)$, and a^{-1} the inverse of $a = d(\sup\{a_n\})$ with respect to "$\circ$". Then we have the following equalities :

$$\begin{aligned} e &= a \circ a^{-1} = d((\sup\{a_n\})\, a^{-1}) = d(\sup\{a_n a^{-1}\}) \\ &= d(\sup(\cup\, L(\{a_n \circ a^{-1}\}). \end{aligned}$$

Since the set $\cup\, L(a_n \circ a^{-1})$ is contained in H, there exists a finite number of $x_i \in L(a_n \circ a^{-1})$ for suitable n with $e = x_1 \vee \cdots \vee x_r$. Then we have $e \leq \sup L(a_n \circ a^{-1})$
all positive integer k. Q. E. D.

An element p in I is said to be prime, if $ab \leq p$ implies $a \leq p$ or $b \leq p$, where a, b are elements in I. An element p in $d(I)$ is said to be $(\circ)$-prime, if $a \circ b \leq p$ implies $a \leq p$ or $b \leq p$ for elements a, b in $d(I)$. Then evidently an element p in $d(I)$ is $(\circ)$-prime, if and only if it is prime.

If S is Artinian, $(d(S), \leq)$ forms a distributive lattice, so that we obtain an abstract refinement theorem for elements of $d(I)$. The proof is quite analogous to Artin's refinement theorem for integral ideals in integral domains [3] or to Theorem 2. 4 in [1]. Consequently we obtain the following :

Theorem 4. *Let S be an Artinian semigroup. If a. c. c. holds for the elements in* $d(I)$, $(d(S), \circ)$ *is a restricted direct product of infinite cyclic groups, each of which is generated by the prime elements in* $d(I)$. *In particular, each element in* $d(I)$ *is uniquely factored, apart from commutativity, into a finite* $(\circ)$*-product of prime elements.*

We say that *arithmetic holds for* $(d(S), \circ)$ or in other words $(S(S), \circ)$ *is arithmetical,* if $d(S)$ has the factorization mentioned in the above theorem. Let a and b be two elements in S. If $d(a) = d(b)$, a is said to be *quasi-equal* to b ; in symbol : $a \sim b$.

Corollary. *If* $(d(S), \circ)$ *is arithmetical, then each element a in S is factored as follows :*

$$a \sim (\Pi\, p^{n(p,\, a)(+)})\ (\Pi\, p^{n(p,\, a)(-)}),$$

where $n(p, a)(+)$ *and* $n(p, a)(-)$ *are positive and negative exponents of a at p.*

Now we introduced *lattice-theoretical divisor theory* in [8]. From this standpoint of view we obtain the following :

Theorem 5. *Let S be a pal-semigroup with a. c. c. for elements in d* (I). *Then the following three conditions are equivalent to one another :*

(1) *A divisor theory for I based on H exists.*

(2) *e is integrally closed.*

(3) *S is Artinian.*

Proof. $(1) \Rightarrow (2)$ is obtained by Theorem 3 in [8]. $(2) \Rightarrow (3)$ is immediate by Theorem 2. $(3) \Rightarrow (1)$: By Theorem 4 and Corollary 2 to Theorem 3 in [8] we can show that $(I, H, d, d(I))$ is a divisor theory of I based on H. Q. E. D.

In the rest of this section, we suppose that $S = (S, \bullet, \leq)$ is a *pal-semigroup.* If $(S, \bullet)$ is a group which is a restricted direct product of infinite cyclic groups with prime generators in I, we say that *arithmetic holds for* S or S *is arithmetical.*

Theorem 6. *If* $(S, \cdot)$ *is a group such that* e *is finitely attainable, then arithmetic holds for* S.

Proof. S is clearly Artinian, so that e is integrally closed by Theorem 4. Moreover $a \sim b$ implies $a = b$, and every prime different from e is divisor-free (maximal) in I. It is readily shown that if T is the join-semi-lattice generated by H in S, then a. c. c. holds for elments in I if and only if $T = I$.

Theorem 7. *Let* S *be an Artinian semigroup. If a. c. c. holds for the elements in* I, *and each prime element in* I *is divisor-free, then* S *is arithmetical.*

Proof. It is clear that the condition (3) of Theorem 2. 6 in [1] is true for our case. Therefore we obtain the result mentioned above. Q. E. D.

Lastly we note that factorization of lattice-ideals in S is obtained, utilizing the results in [4], [5], [6] and [7].

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