Note on recursive formulae for a special case

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Abstract

We calculate the Lescop invariant of every Brieskorn-Hamm manifold by Lescop's surgery formulae. By the result, we give recursive formulae for a special case.

1 Introduction

In 1985, A. Casson [1] defined an invariant for oriented integral homology 3-spheres from representations of their fundamental groups into $SU(2)$. K. Walker [15] extended it to oriented rational homology 3-spheres. C. Lescop [5] gave a formula to calculate the invariant from framed link presentations, and found that the invariant can be extended to all oriented closed 3-manifolds. We call the last invariant the *Lescop invariant*, and denote it by $\lambda(M)$ for an oriented closed 3-manifold M.

Let $a_1, \dots, a_n$ be positive integers where $n \geq 3$, and $B = (b_{ij})$ an $(n-2) \times n$ matrix over the complex number field such that each maximal minor is non-zero (see [3]). Then the variety

$$V_B(a_1, \dots, a_n) = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} b_{ij} z_j^{a_j} = 0 \ (i = 1, \dots, n-2) \right\}$$

is a complex surface which is non-singular except at the origin, and we get

$$\Sigma(a_1, \dots, a_n) = V_B(a_1, \dots, a_n) \cap S^{2n-1}$$

where S^{2n-1} is the boundary of a sufficiently large ball in $\mathbb{C}^n$ including the origin. We call the 3-manifold $\Sigma(a_1, \dots, a_n)$ the *Brieskorn-Hamm manifold*. We note that $\Sigma(a_1, \dots, a_n)$ is a Seifert fibered 3-manifold, the diffeo-type and the orientation of $\Sigma(a_1, \dots, a_n)$ are independent from choices of B and the order of indices of $a_1, \dots, a_n$, and $\Sigma(a_1, \dots, a_n)$ with $n \geq 3$ is an a_n-fold cyclic branched covering of $\Sigma(a_1, \dots, a_{n-1})$ whose branch set is determined by $z_n = 0$ (In particular, if $n = 3$, then the set is an (a_1, a_2)-torus knot/link in S^3) (see [10] and [11]). Since $\Sigma(a_1, a_2, 1) = S^3$ and $\Sigma(a_1, \dots, a_{n-1}, 1) = \Sigma(a_1, \dots, a_{n-1})$ when $n \geq 4$, we may assume $a_i \geq 2$ for all $i = 1, \dots, n$. Throughout this paper, we assume it. We set

$$\lambda(a_1, \dots, a_n) = \lambda\left(\Sigma(a_1, \dots, a_n)\right).$$

We calculate the Lescop invariant of every Brieskorn-Hamm manifold by Lescop's surgery formulae in Theorem 3.2. By the result, we give recursive formulae for a special case as the following theorem.

Theorem 4.1. *We suppose the conditions* (1) *as in Lemma 2.2 and Theorem 3.2. We set* $b = \prod_{i=1}^{n} b_i$.

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(a) *If* $(a_1,\dots,a_n)=(db_1,db_2,b_3,\dots,b_n)$ *satisfies that* $b_1,\dots,b_n$ *are pairwise coprime integers, and* $\gcd(d,b_j)=1$ *for* $j=3,\dots,n$, *then we have the following:*

(i) *We set* $\widehat{b}_1=\dfrac{b}{b_1}$, $b'_1=b_1+k\widehat{b}_1>0$ *for an integer* k, *and* $a'_1=db'_1$. *Then we have*

$$\lambda(a'_1,a_2,\dots,a_n)-\lambda(a_1,a_2,\dots,a_n)$$
$$=-\frac{k}{24}\left(\frac{\widehat{b}_1}{b_2}\right)^{d-1}\left\{1-\left(\frac{\widehat{b}_1}{b_2}\right)^2+d\widehat{b}_1^2\left(n-2-\sum_{j=3}^{n}\frac{1}{b_j^2}\right)\right\}.$$

(ii) *We set* $\widehat{b}_3=\dfrac{b}{b_3}$, $b'_3=b_3+dk\widehat{b}_3>0$ *for an integer* k, *and* $a'_3=b'_3$. *Then we have*

$$\frac{\lambda(a_1,a_2,a'_3,a_4,\dots,a_n)}{b_3'^{d-1}}-\frac{\lambda(a_1,a_2,a_3,a_4,\dots,a_n)}{b_3^{d-1}}$$
$$=-\frac{dk}{24}\left(\frac{\widehat{b}_3}{b_1b_2}\right)^{d-1}\left\{1-\left(\frac{\widehat{b}_3}{b_1}\right)^2-\left(\frac{\widehat{b}_3}{b_2}\right)^2-\frac{d\widehat{b}_3^2(d-1)}{b_3b'_3}+d\widehat{b}_3^2\left(n-2-\sum_{j=4}^{n}\frac{1}{b_j^2}\right)\right\}.$$

(b) *If* $(a_1,\dots,a_n)=(2b_1,2b_2,2b_3,b_4,\dots,b_n)$ *satisfies that* $b_1,\dots,b_n$ *are pairwise coprime integers, and* $\gcd(2,b_j)=1$ *for* $j=4,\dots,n$, *then we have the following:*

(i) *We set* $\widehat{b}_1=\dfrac{b}{b_1}$, $b'_1=b_1+k\widehat{b}_1>0$ *for an integer* k, *and* $a'_1=2b'_1$. *Then we have*

$$\frac{\lambda(a'_1,a_2,\dots,a_n)}{b'_1}-\frac{\lambda(a_1,a_2,\dots,a_n)}{b_1}$$
$$=-\frac{k\widehat{b}_1^3}{12b_2^2b_3^2}\left\{2-\left(\frac{\widehat{b}_1}{b_2}\right)^2-\left(\frac{\widehat{b}_1}{b_3}\right)^2-\frac{\widehat{b}_1^2}{b_1b'_1}+2\widehat{b}_1^2\left(n-2-\sum_{j=4}^{n}\frac{1}{b_j^2}\right)\right\}.$$

(ii) *We set* $\widehat{b}_4=\dfrac{b}{b_4}$, $b'_4=b_4+2k\widehat{b}_4>0$ *for an integer* k, *and* $a'_4=b'_4$. *Then we have*

$$\frac{\lambda(a_1,a_2,a_3,a'_4,a_5,\dots,a_n)}{b_4'^3}-\frac{\lambda(a_1,a_2,a_3,a_4,a_5,\dots,a_n)}{b_4^3}$$
$$=-\frac{k\widehat{b}_4^3}{6b_1^2b_2^2b_3^2}\left\{2-\left(\frac{\widehat{b}_4}{b_1}\right)^2-\left(\frac{\widehat{b}_4}{b_2}\right)^2-\left(\frac{\widehat{b}_4}{b_3}\right)^2-\frac{6\widehat{b}_4^2}{b_4b'_4}+2\widehat{b}_4^2\left(n-2-\sum_{j=5}^{n}\frac{1}{b_j^2}\right)\right\}.$$

In Section 2, we introduce a surgery description of the Brieskorn-Hamm manifolds. In Section 3, we compute the Lescop invariant of every Brieskorn-Hamm manifold. In Section 4, we show recursive formulae for a special case which is an extension of a result due to S. Fukuhara, Y. Matsumoto and K. Sakamoto [2, Theorem 4], and independently, W. Neumann and J. Wahl [9, Remark 1.15].

2 Surgery description and the first homology of the Brieskorn-Hamm manifold

In this section, we give a surgery description of the Brieskorn-Hamm manifolds. We denote an oriented Seifert fibered 3-manifold with m-singular fibers whose base space is an oriented closed surface with genus g by

$$(g\mid h;(\alpha_1,\beta_1),\dots,(\alpha_m,\beta_m))$$

where h is the obstruction class, $\gcd(\alpha_i, \beta_i) = 1$ $(i = 1, \dots, m)$, and $\alpha_i \neq 0$ and (α_i, β_i) are the multiplicity and the index of the i-th singular fiber, respectively.

We set

$$s_j = \frac{\prod_{k \neq j} a_k}{\mathrm{lcm}_{k \neq j} a_k}, \quad t_j = \frac{\mathrm{lcm}_k a_k}{\mathrm{lcm}_{k \neq j} a_k} \quad (j = 1, \dots, n), \tag{2.1}$$

and

$$g = \frac{1}{2}\left(2 + \frac{(n-2)\prod_{k=1}^{n} a_k}{\mathrm{lcm}_k a_k} - \sum_{j=1}^{n} s_j\right) \tag{2.2}$$

where lcm denotes the least common multiple. W. Neumann and F. Raymond [8] showed the following:

Lemma 2.1. The Brieskorn-Hamm manifold $\Sigma(a_1, \dots, a_n)$ is presented by

$$(g \mid 0; s_1(t_1, c_1), \dots, s_n(t_n, c_n))$$

where s_j and t_j $(j = 1, \dots, n)$ are in (2.1), $s_j(t_j, c_j)$ implies that (t_j, c_j) is repeated s_j times, c_j satisfies the equation

$$\sum_{j=1}^{n} \frac{s_j}{t_j} c_j = \frac{\prod_{k=1}^{n} a_k}{(\mathrm{lcm}_k a_k)^2}, \tag{2.3}$$

and g is in (2.2).

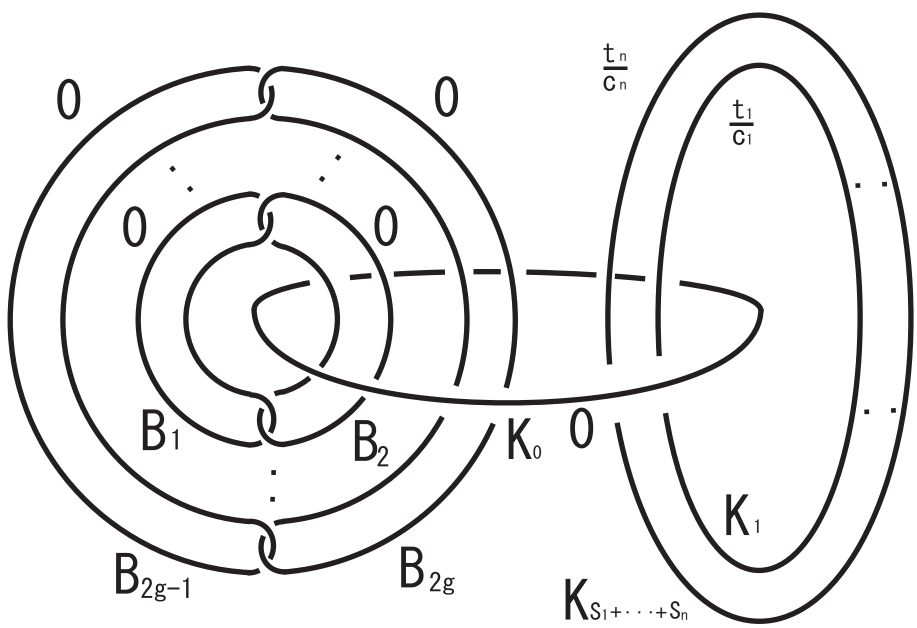

Figure 1: Surgery description of $\Sigma(a_1, \dots, a_n)$

The Brieskorn-Hamm manifold $\Sigma(a_1, \dots, a_n)$ has a surgery description $B_1 \cup \cdots \cup B_{2g} \cup K_0 \cup K_1 \cup \cdots \cup K_{s_1 + \cdots + s_n}$ as in Figure 1 (see [6]). We restrict to the case $g = 0$ and $g = 1$ (see Lemma 3.1 and Teorem 3.2). By [7] and [13], we have the following:

Lemma 2.2. (1) $g = 0$ if and only if one of the following (a) and (b) holds:

(a) By a suitable permutation of $(a_1, \ldots, a_n)$, we have that $a_1/d, a_2/d, a_3, \ldots, a_n$ are pairwise coprime for $d = \gcd(a_1, a_2) \geq 1$, and $\gcd(d, a_j) = 1$ for $j = 3, \ldots, n$.

(b) By a suitable permutation of $(a_1, \ldots, a_n)$, we have that $2 = \gcd(a_1, a_2, a_3)$, $a_1/2, a_2/2, a_3/2, a_4, \ldots, a_n$ are pairwise coprime, and $\gcd(2, a_j) = 1$ for $j = 4, \ldots, n$.

(2) $g = 1$ if and only if one of the following (a), (b), (c) and (d) holds:

(a) By a suitable permutation of $(a_1, \ldots, a_n)$, we have that $a_1/2, a_2/3$ and $a_3/6$ are integers, $a_1/2, a_2/3, a_3/6, a_4, \ldots, a_n$ are pairwise coprime, 6 divides neither a_1 nor a_2, and $\gcd(6, a_j) = 1$ for $j = 4, \ldots, n$.

(b) By a suitable permutation of $(a_1, \ldots, a_n)$, we have that $a_1/2, a_2/4$ and $a_3/4$ are integers, $a_1/2, a_2/4, a_3/4, a_4, \ldots, a_n$ are pairwise coprime, 4 does not divide a_1, and $\gcd(2, a_j) = 1$ for $j = 4, \ldots, n$.

(c) By a suitable permutation of $(a_1, \ldots, a_n)$, we have that $a_1/3, a_2/3$ and $a_3/3$ are integers, $a_1/3, a_2/3, a_3/3, a_4, \ldots, a_n$ are pairwise coprime, and $\gcd(3, a_j) = 1$ for $j = 4, \ldots, n$.

(d) By a suitable permutation of $(a_1, \ldots, a_n)$, we have that $a_1/2, a_2/2, a_3/2$ and $a_4/2$ are integers, $a_1/2, a_2/2, a_3/2, a_4/2, a_5, \ldots, a_n$ are pairwise coprime, and $\gcd(2, a_j) = 1$ for $j = 5, \ldots, n$.

3 Lescop's surgery formulae

We compute the Lescop invariant of $\Sigma(a_1, \ldots, a_n)$ by Lescop's surgery formulae.

Let p be a non-zero integer, and q an integer. For a real number x, we denote by

$$((x)) = \begin{cases} 0 & (x \in \mathbb{Z}), \\ x - [x] - \frac{1}{2} & (x \in \mathbb{R} \setminus \mathbb{Z}), \end{cases}$$

where $[\cdot]$ is the gaussian symbol. Then the Dedekind sum $s(q, p)$ is defined by

$$s(q, p) = \sum_{i=1}^{|p|} \left(\left(\frac{i}{p} \right) \right) \left(\left(\frac{qi}{p} \right) \right).$$

For a real number y, we denote by

$$\varepsilon(y) = \begin{cases} \frac{y}{|y|} & (y \neq 0), \\ -1 & (y = 0). \end{cases}$$

Lemma 3.1. (C. Lescop [5, Proposition 6.1.1]) Let $M = (g \mid h; (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m))$ be an oriented Seifert fibered 3-manifold as in Section 2, and $e = -h + \displaystyle\sum_{i=1}^{m} \frac{\beta_i}{\alpha_i}$ the Seifert invariant of M.

(1) If $g = 0$, then we have

$$\lambda(M) = \prod_{i=1}^{m} \alpha_i \left\{ \frac{\varepsilon(e)}{24} \left(2 - m + \sum_{i=1}^{m} \frac{1}{{\alpha_i}^2} \right) + \frac{|e|e}{24} - \frac{e}{8} - \frac{|e|}{2} \sum_{i=1}^{m} s(\beta_i, \alpha_i) \right\}.$$

(2) If $g = 1$, then we have

$$\lambda(M) = -\varepsilon(e) \prod_{i=1}^{m} \alpha_i.$$

(3) If $g \geq 2$, then we have

$$\lambda(M) = 0.$$

By Lemma 2.2 and Lemma 3.1, we have the following:

Theorem 3.2. We suppose the same settings as in Section 2. We set $b = \prod_{i=1}^{n} b_i$, and c_j ($j = 1, \cdots, n$) satisfies the equation (2.3) in Lemma 2.1.

(1) The case $g = 0$.

(a) If $(a_1, \ldots, a_n) = (db_1, db_2, b_3, \ldots, b_n)$ satisfies that $b_1, \ldots, b_n$ are pairwise coprime integers, and $\gcd(d, b_j) = 1$ for $j = 3, \ldots, n$, then we have

$$\lambda(db_1, db_2, b_3, \ldots, b_n) = \left(\frac{b}{b_1 b_2}\right)^{d-1} \left\{ \frac{b}{24}\left(2d - nd + \sum_{j=1}^{2} \frac{1}{{b_j}^2} + \sum_{j=3}^{n} \frac{d}{{b_j}^2}\right) + \frac{1}{24b} - \frac{1}{8} \right.$$
$$\left. -\frac{1}{2}\sum_{j=1}^{2} s(c_j, b_j) - \frac{d}{2}\sum_{j=3}^{n} s(c_j, b_j) \right\}.$$

(b) If $(a_1, \ldots, a_n) = (2b_1, 2b_2, 2b_3, b_4, \ldots, b_n)$ satisfies that $b_1, \ldots, b_n$ are pairwise coprime integers, and $\gcd(2, b_j) = 1$ for $j = 4, \ldots, n$, then we have

$$\lambda(2b_1, 2b_2, 2b_3, b_4, \ldots, b_n) = \frac{b^3}{{b_1}^2{b_2}^2{b_3}^2} \left\{ \frac{b}{24}\left(8 - 4n + \sum_{j=1}^{3} \frac{2}{{b_j}^2} + \sum_{j=4}^{n} \frac{4}{{b_j}^2}\right) + \frac{1}{6b} - \frac{1}{4} \right.$$
$$\left. -2\sum_{j=1}^{3} s(c_j, b_j) - 4\sum_{j=4}^{n} s(c_j, b_j) \right\}.$$

(2) The case $g = 1$.

(a) If $(a_1, \ldots, a_n) = (2b_1, 3b_2, 6b_3, b_4, \ldots, b_n)$ satisfies that $b_1, \ldots, b_n$ are pairwise coprime integers, 6 divides neither a_1 nor a_2, and $\gcd(6, b_j) = 1$ for $j = 4, \ldots, n$, then we have

$$\lambda(2b_1, 3b_2, 6b_3, b_4, \ldots, b_n) = -\frac{b^6}{{b_1}^3{b_2}^4{b_3}^5}.$$

(b) If $(a_1, \ldots, a_n) = (2b_1, 4b_2, 4b_3, b_4, \ldots, b_n)$ satisfies that $b_1, \ldots, b_n$ are pairwise coprime integers, 4 does not divide a_1, and $\gcd(2, b_j) = 1$ for $j = 4, \ldots, n$, then we have

$$\lambda(2b_1, 4b_2, 4b_3, b_4, \ldots, b_n) = -\frac{b^8}{{b_1}^4{b_2}^6{b_3}^6}.$$

(c) If $(a_1, \ldots, a_n) = (3b_1, 3b_2, 3b_3, b_4, \ldots, b_n)$ satisfies that $b_1, \ldots, b_n$ are pairwise coprime integers, and $\gcd(3, b_j) = 1$ for $j = 4, \ldots, n$, then we have

$$\lambda(3b_1, 3b_2, 3b_3, b_4, \ldots, b_n) = -\frac{b^9}{{b_1}^6{b_2}^6{b_3}^6}.$$

(d) If $(a_1, \ldots, a_n) = (2b_1, 2b_2, 2b_3, 2b_4, b_5, \ldots, b_n)$ satisfies that $b_1, \ldots, b_n$ are pairwise coprime integers, and $\gcd(2, b_j) = 1$ for $j = 5, \ldots, n$, then we have

$$\lambda(2b_1, 2b_2, 2b_3, 2b_4, b_5, \ldots, b_n) = -\frac{b^8}{{b_1}^4{b_2}^4{b_3}^4{b_4}^4}.$$

(3) The case $g \geq 2$.

$$\lambda(a_1, \ldots, a_n) = 0.$$

We remark that Theorem 3.2 holds even if $a_i = 1$ is included.

4 Recursive formulae for a special case

In this section, we give recursive formulae for a special case.

Theorem 4.1. We suppose the conditions (1) as in Lemma 2.2 and Theorem 3.2. We set $b = \prod_{i=1}^{n} b_i$.

(a) If $(a_1, \ldots, a_n) = (db_1, db_2, b_3, \ldots, b_n)$ satisfies that $b_1, \ldots, b_n$ are pairwise coprime integers, and $\gcd(d, b_j) = 1$ for $j = 3, \ldots, n$, then we have the following:

(i) We set $\widehat{b_1} = \dfrac{b}{b_1}$, $b_1' = b_1 + k\widehat{b_1} > 0$ for an integer k, and $a_1' = db_1'$. Then we have

$$\begin{aligned}
&\lambda(a_1', a_2, \ldots, a_n) - \lambda(a_1, a_2, \ldots, a_n) \\
&= -\frac{k}{24}\left(\frac{\widehat{b_1}}{b_2}\right)^{d-1}\left\{1 - \left(\frac{\widehat{b_1}}{b_2}\right)^2 + d\widehat{b}_1^2\left(n - 2 - \sum_{j=3}^{n}\frac{1}{b_j^2}\right)\right\}.
\end{aligned}$$

(ii) We set $\widehat{b_3} = \dfrac{b}{b_3}$, $b_3' = b_3 + dk\widehat{b_3} > 0$ for an integer k, and $a_3' = b_3'$. Then we have

$$\begin{aligned}
&\frac{\lambda(a_1, a_2, a_3', a_4, \ldots, a_n)}{b_3'^{d-1}} - \frac{\lambda(a_1, a_2, a_3, a_4, \ldots, a_n)}{b_3^{d-1}} \\
&= -\frac{dk}{24}\left(\frac{\widehat{b_3}}{b_1 b_2}\right)^{d-1}\left\{1 - \left(\frac{\widehat{b_3}}{b_1}\right)^2 - \left(\frac{\widehat{b_3}}{b_2}\right)^2 - \frac{d\widehat{b}_3^2(d-1)}{b_3 b_3'} + d\widehat{b}_3^2\left(n - 2 - \sum_{j=4}^{n}\frac{1}{b_j^2}\right)\right\}.
\end{aligned}$$

(b) If $(a_1, \ldots, a_n) = (2b_1, 2b_2, 2b_3, b_4, \ldots, b_n)$ satisfies that $b_1, \ldots, b_n$ are pairwise coprime integers, and $\gcd(2, b_j) = 1$ for $j = 4, \ldots, n$, then we have the following:

(i) We set $\widehat{b}_1 = \dfrac{b}{b_1}$, $b'_1 = b_1 + k\widehat{b}_1 > 0$ for an integer k, and $a'_1 = 2b'_1$. Then we have

$$\frac{\lambda(a'_1, a_2, \ldots, a_n)}{b'_1} - \frac{\lambda(a_1, a_2, \ldots, a_n)}{b_1}$$
$$= -\frac{k\widehat{b}_1^3}{12b_2^2b_3^2}\left\{2 - \left(\frac{\widehat{b}_1}{b_2}\right)^2 - \left(\frac{\widehat{b}_1}{b_3}\right)^2 - \frac{\widehat{b}_1^2}{b_1b'_1} + 2\widehat{b}_1^2\left(n - 2 - \sum_{j=4}^{n}\frac{1}{b_j^2}\right)\right\}.$$

(ii) We set $\widehat{b}_4 = \dfrac{b}{b_4}$, $b'_4 = b_4 + 2k\widehat{b}_4 > 0$ for an integer k, and $a'_4 = b'_4$. Then we have

$$\frac{\lambda(a_1, a_2, a_3, a'_4, a_5, \ldots, a_n)}{b'^3_4} - \frac{\lambda(a_1, a_2, a_3, a_4, a_5, \ldots, a_n)}{b_4^3}$$
$$= -\frac{k\widehat{b}_4^3}{6b_1^2b_2^2b_3^2}\left\{2 - \left(\frac{\widehat{b}_4}{b_1}\right)^2 - \left(\frac{\widehat{b}_4}{b_2}\right)^2 - \left(\frac{\widehat{b}_4}{b_3}\right)^2 - \frac{6\widehat{b}_4^2}{b_4b'_4} + 2\widehat{b}_4^2\left(n - 2 - \sum_{j=5}^{n}\frac{1}{b_j^2}\right)\right\}.$$

The case $d = 1$ of Theorem 4.1 (a) (i) corresponds to a theorem due to S. Fukuhara, Y. Matsumoto and K. Sakamoto [2]. We remark that the righthand side of (a) (i) is independent from b_1, and the author [12] and Yukihiro Tsutsumi [14] showed this kind formulae for branched cyclic coverings of S^3 over some satellite knots. We need the following lemmas to prove Theorem 4.1.

Lemma 4.2. Let p be a positive integer, and q an integer which is coprime to p. Then we have:

(1) $s(q + np, p) = s(q, p)$ $(n \in \mathbb{Z})$, $s(-q, p) = -s(q, p)$, $s(\bar{q}, p) = s(q, p)$ where $q\bar{q} \equiv 1 \pmod{p}$.

(2) ([4]) $s(q, p) + s(p, q) = \dfrac{p^2 + q^2 + 1 - 3pq}{12pq}$ $\quad (p, q > 0)$.

Lemma 4.3. We suppose the conditions as in Theorem 4.1. We set $b'_j = b_j$ for the rest indices in every case of Theorem 4.1, and $b' = \prod_{i=1}^{n} b'_i$. Let c_j $(j = 1, \ldots, n)$ be the same as in (2.3) corresponding to $(a_1, \ldots, a_n)$, and c'_j $(j = 1, \ldots, n)$ the same as c_j in (2.3) corresponding to $(a'_1, a_2, \ldots, a_n)$, $(a_1, a_2, a'_3, a_4, \ldots, a_n)$, or $(a_1, a_2, a_3, a'_4, a_5, \ldots, a_n)$, respectively.

(a) Suppose the condition as in Theorem 4.1 (a). Then we have

$$\begin{aligned} \frac{b}{b_1}c_1 + \frac{b}{b_2}c_2 + d\sum_{j=3}^{n}\frac{b}{b_j}c_j &= 1 \\ \textit{and} & \\ \frac{b'}{b'_1}c'_1 + \frac{b'}{b'_2}c'_2 + d\sum_{j=3}^{n}\frac{b'}{b'_j}c'_j &= 1. \end{aligned} \tag{4.1}$$

(i) Suppose the condition as in Theorem 4.1 (a) (i). Then we have

$$\widehat{b}_1 = \frac{b}{b_1} = \frac{b'}{b'_1} = \prod_{i=2}^{n} b_i, \quad \widehat{b}_1 c_1 \equiv 1 \pmod{b_1}, \quad \widehat{b}_1 c'_1 \equiv 1 \pmod{b'_1},$$

and

$$c_j \equiv c'_j \pmod{b_j} \quad (j = 2, \ldots, n).$$

(ii) Suppose the condition as in Theorem 4.1 (a) (ii). Then we have

$$\widehat{b}_3 = \frac{b}{b_3} = \frac{b'}{b'_3} = \prod_{\substack{1\le i\le n\\ i\neq 3}} b_i, \quad d\widehat{b}_3 c_3 \equiv 1 \pmod{b_3}, \quad d\widehat{b}_3 c'_3 \equiv 1 \pmod{b'_3},$$

and

$$c_j \equiv c'_j \pmod{b_j} \quad (j = 1, 2, 4, \ldots, n).$$

(b) Suppose the condition as in Theorem 4.1 (b). Then we have

$$\begin{aligned} \frac{b}{b_1}c_1 + \frac{b}{b_2}c_2 + \frac{b}{b_3}c_3 + 2\sum_{j=4}^{n} \frac{b}{b_j}c_j &= 1 \\ \text{and} & \\ \frac{b'}{b'_1}c'_1 + \frac{b'}{b'_2}c'_2 + \frac{b'}{b'_3}c'_3 + 2\sum_{j=4}^{n} \frac{b'}{b'_j}c'_j &= 1. \end{aligned} \tag{4.2}$$

(i) Suppose the condition as in Theorem 4.1 (b) (i). Then we have

$$\widehat{b}_1 = \frac{b}{b_1} = \frac{b'}{b'_1} = \prod_{i=2}^{n} b_i, \quad \widehat{b}_1 c_1 \equiv 1 \pmod{b_1}, \quad \widehat{b}_1 c'_1 \equiv 1 \pmod{b'_1},$$

and

$$c_j \equiv c'_j \pmod{b_j} \quad (j = 2, \ldots, n).$$

(ii) Suppose the condition as in Theorem 4.1 (b) (ii). Then we have

$$\widehat{b}_4 = \frac{b}{b_4} = \frac{b'}{b'_4} = \prod_{\substack{1\le i\le n\\ i\neq 4}} b_i, \quad 2\widehat{b}_4 c_4 \equiv 1 \pmod{b_4}, \quad 2\widehat{b}_4 c'_4 \equiv 1 \pmod{b'_4},$$

and

$$c_j \equiv c'_j \pmod{b_j} \quad (j = 1, 2, 3, 5, \ldots, n).$$

Proof. (a) By Lemma 2.1, we have the result.

(i) Since b/b_j and b'/b'_j are divisible by b_i and b'_i for $i \neq j$, respectively, and (4.1), we have the result.

(ii) In the similar way as (i), we have the result.

(b) By Lemma 2.1, we have the result.

(i) Since b/b_j and b'/b'_j are divisible by b_i and b'_i for $i \neq j$, respectively, and (4.2), we have the result.

(ii) In the similar way as (i), we have the result. □

Lemma 4.4. We suppose the conditions as in Lemma 4.3.

(a) (i) Suppose the condition as in Lemma 4.3 (a) (i). Then we have

$$s(c'_1, b'_1) - s(c_1, b_1) = -\frac{k}{12 b_1 b'_1}\left(\widehat{b}_1^2 - b_1 b'_1 + 1\right),$$

and

$$s(c_j, b_j) = s(c'_j, b'_j) \quad (j = 2, \ldots, n).$$

(ii) Suppose the condition as in Lemma 4.3 (a) (ii). Then we have

$$s(c_3', b_3') - s(c_3, b_3) = -\frac{k}{12b_3b_3'}\left(d^2\widehat{b}_3^2 - b_3b_3' + 1\right),$$

and

$$s(c_j, b_j) = s(c_j', b_j') \quad (j = 1, 2, 4, \ldots, n).$$

(b) (i) Suppose the condition as in Lemma 4.3 (b) (i). Then we have

$$s(c_1', b_1') - s(c_1, b_1) = -\frac{k}{12b_1b_1'}\left(\widehat{b}_1^2 - b_1b_1' + 1\right),$$

and

$$s(c_j, b_j) = s(c_j', b_j') \quad (j = 2, \ldots, n).$$

(ii) Suppose the condition as in Lemma 4.3 (b) (ii). Then we have

$$s(c_4', b_4') - s(c_4, b_4) = -\frac{k}{12b_4b_4'}\left(4\widehat{b}_4^2 - b_4b_4' + 1\right),$$

and

$$s(c_j, b_j) = s(c_j', b_j') \quad (j = 1, 2, 3, 5, \ldots, n).$$

Proof. (a) (i) By Lemma 4.2 (1), Lemma 4.2 (2) and Lemma 4.3 (a) (i), we have

$$\begin{aligned} s(c_1, b_1) &= s\left(\widehat{b}_1, b_1\right) \\ &= \frac{\widehat{b}_1^2 + b_1^2 + 1 - 3\widehat{b}_1b_1}{12b_1\widehat{b}_1} - s(b_1, \widehat{b}_1), \\ s(c_1', b_1') &= s\left(\widehat{b}_1, b_1'\right) \\ &= \frac{\widehat{b}_1^2 + b_1'^2 + 1 - 3\widehat{b}_1b_1'}{12b_1'\widehat{b}_1} - s(b_1 + k\widehat{b}_1, \widehat{b}_1) \\ &= \frac{\widehat{b}_1^2 + b_1'^2 + 1 - 3\widehat{b}_1b_1'}{12b_1'\widehat{b}_1} - s(b_1, \widehat{b}_1), \end{aligned}$$

and

$$\begin{aligned} s(c_1', b_1') - s(c_1, b_1) &= \frac{\widehat{b}_1^2 + b_1'^2 + 1 - 3\widehat{b}_1b_1'}{12b_1'\widehat{b}_1} - \frac{\widehat{b}_1^2 + b_1^2 + 1 - 3\widehat{b}_1b_1}{12b_1\widehat{b}_1} \\ &= \frac{1}{12\widehat{b}_1}\left\{\left(\widehat{b}_1^2 + 1\right)\left(\frac{1}{b_1'} - \frac{1}{b_1}\right) + (b_1' - b_1)\right\} \\ &= \frac{k}{12\widehat{b}_1}\left\{-\frac{\widehat{b}_1}{b_1b_1'}\left(\widehat{b}_1^2 + 1\right) + \widehat{b}_1\right\} \\ &= -\frac{k}{12b_1b_1'}\left(\widehat{b}_1^2 - b_1b_1' + 1\right). \end{aligned}$$

By Lemma 4.3 (a) (i), we have $s(c_j, b_j) = s(c_j', b_j')$ for $j \neq 1$.

(ii) By Lemma 4.2 (1), Lemma 4.2 (2) and Lemma 4.3 (a) (ii), we have

$$\begin{aligned} s(c_3, b_3) &= s\left(d\widehat{b}_3, b_3\right) \\ &= \frac{d^2\widehat{b}_3^2 + b_3^2 + 1 - 3d\widehat{b}_3 b_3}{12db_3\widehat{b}_3} - s(b_3, d\widehat{b}_3), \\ s(c_3', b_3') &= s\left(d\widehat{b}_3, b_3'\right) \\ &= \frac{d^2\widehat{b}_3^2 + b_3'^2 + 1 - 3d\widehat{b}_3 b_3'}{12db_3'\widehat{b}_3} - s(b_3 + dk\widehat{b}_3, d\widehat{b}_3) \\ &= \frac{d^2\widehat{b}_3^2 + b_3'^2 + 1 - 3d\widehat{b}_3 b_3'}{12db_3'\widehat{b}_3} - s(b_3, d\widehat{b}_3), \end{aligned}$$

and

$$\begin{aligned} s(c_3', b_3') - s(c_3, b_3) &= \frac{d^2\widehat{b}_3^2 + b_3'^2 + 1 - 3d\widehat{b}_3 b_3'}{12db_3'\widehat{b}_3} - \frac{d^2\widehat{b}_3^2 + b_3^2 + 1 - 3d\widehat{b}_3 b_3}{12db_3\widehat{b}_3} \\ &= \frac{1}{12d\widehat{b}_3}\left\{\left(d^2\widehat{b}_3^2 + 1\right)\left(\frac{1}{b_3'} - \frac{1}{b_3}\right) + (b_3' - b_3)\right\} \\ &= \frac{k}{12\widehat{b}_3}\left\{-\frac{\widehat{b}_3}{b_3 b_3'}\left(d^2\widehat{b}_3^2 + 1\right) + \widehat{b}_3\right\} \\ &= -\frac{k}{12 b_3 b_3'}\left(d^2\widehat{b}_3^2 - b_3 b_3' + 1\right). \end{aligned}$$

By Lemma 4.3 (a) (ii), we have $s(c_j, b_j) = s(c_j', b_j')$ for $j \neq 3$.

(b) By the similar way as (a), we have the result. □

Proof of Theorem 4.1. (a) (i) We set

$$A' = 2d - nd + \frac{1}{{b_1'}^2} + \frac{1}{{b_2}^2} + \sum_{j=3}^{n} \frac{d}{{b_j}^2},\ B' = -\frac{1}{2}s\left(c_1', b_1'\right) - \frac{1}{2}s\left(c_2', b_2'\right) - \frac{d}{2}\sum_{j=3}^{n} s\left(c_j', b_j'\right),$$

$$A = 2d - nd + \sum_{j=1}^{2}\frac{1}{{b_j}^2} + \sum_{j=3}^{n}\frac{d}{{b_j}^2},\ \text{and}\ B = -\frac{1}{2}\sum_{j=1}^{2} s(c_j, b_j) - \frac{d}{2}\sum_{j=3}^{n} s(c_j, b_j).$$

Then we have

$$b_1'\widehat{b}_1 A' - bA = \left\{-\frac{k\widehat{b}_1^2}{b_1 b_1'} + \frac{k\widehat{b}_1^2}{{b_2}^2} - dk\widehat{b}_1^2\left(n - 2 - \sum_{j=3}^{n}\frac{1}{{b_j}^2}\right)\right\}.$$

By Lemma 4.3 (a) (i) and Lemma 4.4 (a) (i), we have

$$\begin{aligned} 24B' - 24B &= -12s\left(c_1', b_1'\right) + 12s(c_1, b_1) \\ &= \frac{k}{b_1 b_1'}\left(\widehat{b}_1^2 - b_1 b_1' + 1\right). \end{aligned}$$

By Theorem 3.2 (1) (a), we have

$$\begin{aligned}
&\lambda(a_1', a_2, \dots, a_n) - \lambda(a_1, a_2, \dots, a_n) \\
&= \lambda\left(db_1{}', db_2, b_3, \dots, b_n\right) - \lambda(db_1, db_2, b_3 \dots, b_n) \\
&= \left(\frac{b_1'\widehat{b}_1}{b_1'b_2}\right)^{d-1}\left(\frac{b_1'\widehat{b}_1}{24}A' + \frac{1}{24b_1'\widehat{b}_1} - \frac{1}{8} + B'\right) - \left(\frac{b}{b_1b_2}\right)^{d-1}\left(\frac{b}{24}A + \frac{1}{24b} - \frac{1}{8} + B\right) \\
&= \frac{1}{24}\left(\frac{\widehat{b}_1}{b_2}\right)^{d-1}\left(b_1'\widehat{b}_1A' - bA - \frac{k\widehat{b}_1}{b_1b_1'} + 24B' - 24B\right) \\
&= -\frac{k}{24}\left(\frac{\widehat{b}_1}{b_2}\right)^{d-1}\left\{1 - \left(\frac{\widehat{b}_1}{b_2}\right)^2 + d\widehat{b}_1^2\left(n - 2 - \sum_{j=3}^{n}\frac{1}{b_j^2}\right)\right\}.
\end{aligned}$$

(ii) We set

$$C' = 2d - nd + \sum_{j=1}^{2}\frac{1}{b_j{}^2} + \frac{d}{b_3'{}^2} + \sum_{j=4}^{n}\frac{d}{b_j{}^2},\ D' = -\sum_{j=1}^{2}\frac{1}{2}s\left(c_j', b_j'\right) - \frac{d}{2}s\left(c_3', b_3'\right) - \frac{d}{2}\sum_{j=4}^{n}s\left(c_j', b_j'\right),$$

$$C = 2d - nd + \sum_{j=1}^{2}\frac{1}{b_j{}^2} + \sum_{j=3}^{n}\frac{d}{b_j{}^2},\ \text{and } D = -\frac{1}{2}\sum_{j=1}^{2}s(c_j, b_j) - \frac{d}{2}\sum_{j=3}^{n}s(c_j, b_j).$$

Then we have

$$b_3'\widehat{b}_3C' - bC = \left\{-\frac{d^2k\widehat{b}_3^2}{b_3b_3'} + dk\widehat{b}_3^2\sum_{j=1}^{2}\frac{1}{b_j{}^2} - d^2k\widehat{b}_3^2\left(n - 2 - \sum_{j=4}^{n}\frac{1}{b_j{}^2}\right)\right\}.$$

By Lemma 4.3 (a) (ii) and Lemma 4.4 (a) (ii), we have

$$\begin{aligned}
24B' - 24B &= -12ds\left(c_3', b_3'\right) + 12ds(c_3, b_3) \\
&= \frac{dk}{b_3b_3'}\left(d^2\widehat{b}_3^2 - b_3b_3' + 1\right).
\end{aligned}$$

By Theorem 3.2 (1) (b), we have

$$\begin{aligned}
&\frac{\lambda(a_1, a_2, a_3', a_4, \dots, a_n)}{b_3'^{d-1}} - \frac{\lambda(a_1, a_2, a_3, a_4, \dots, a_n)}{b_3^{d-1}} \\
&= \frac{\lambda\left(db_1, db_2, b_3', b_4, \dots, b_n\right)}{b_3'^{d-1}} - \frac{\lambda(db_1, db_2, b_3, b_4, \dots, b_n)}{b_3^{d-1}} \\
&= \frac{1}{b_3'^{d-1}}\left(\frac{b_3'\widehat{b}_3}{b_1b_2}\right)^{d-1}\left(\frac{b_3'\widehat{b}_3}{24}C' + \frac{1}{24b_3'\widehat{b}_3} - \frac{1}{8} + D'\right) - \frac{1}{b_3^{d-1}}\left(\frac{b}{b_1b_2}\right)^{d-1}\left(\frac{b}{24}C + \frac{1}{24b} - \frac{1}{8} + D\right) \\
&= \frac{1}{24}\left(\frac{\widehat{b}_3}{b_1b_2}\right)^{d-1}\left(b_3'\widehat{b}_3C' - bC - \frac{dk}{b_3b_3'} + 24D' - 24D\right) \\
&= -\frac{dk}{24}\left(\frac{\widehat{b}_3}{b_1b_2}\right)^{d-1}\left\{1 - \left(\frac{\widehat{b}_3}{b_1}\right)^2 - \left(\frac{\widehat{b}_3}{b_2}\right)^2 - \frac{d\widehat{b}_3^2(d-1)}{b_3b_3'} + d\widehat{b}_3^2\left(n - 2 - \sum_{j=4}^{n}\frac{1}{b_j^2}\right)\right\}.
\end{aligned}$$

(b) By the similar way as (a), we have the result. □

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