On K-theory of Certain Module Algebra

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Abstract

Let H be a finite dimensional Hopf algebra over a field and A an H-module algebra. This paper starts by comparing the K-theory and the G-theory of a certain relative category of left A#H-modules given by homological isomorphisms. This paper also disputes a condition to give an equivalence between them.

Keywords: *K-theory, Hopf algebra, Module algebra*

1 Introduction

Let Λ(d) be the exterior algebra generated by a degree (-1) element d. The algebra Λ(d) is actually a Hopf algebra, and there is a category equivalence between the category of chain complexes and the category of Z-graded Λ(d)-modules. Therefore, the category of modules over a Hopf algebra is a generalization of the category of chain complexes.

The category of chain complexes is endowed with two types of model structure; the relative model structure and the projective model structure.

Let H be a finite dimensional Hopf algebra over a field k and A an H-module algebra. In Qi's paper[(4)], he defined acyclic objects and quasi-isomorphisms by using null-homotopy and contractible objects, i.e., projective or injective objects, which seems an analogue of the relative model structure rather than the projective model structure on the category of chain complexes. Note that a model structure of the category of A#H-modules is not given in Qi[(4)].

On the other hand, in, Tamaki and the author[(3)] gave the projective model structure on the category of A#H-modules by using homological isomorphisms.

In this paper, we dispute some conditions that is necessary to compare the K-theory and G-theory arising from the projective model structure of left modules over an H-module algebra.

Let k be a field. Let H be a finite-dimensional Hopf algebra which is not semisimple. Let A be a left H-module k-algebra.

We denote by LMod(A#H) the category of left A#H-modules, which is category equivalent to the category of H-equivariant A-modules.

We define qCof and qAcyc by the subcategory of LMod(A#H) consisting of q-cofibrants and those objects q-equivalent to zero, respectively. Here, q-equivalences are given by homological isomorphisms as in Tamaki and the author's paper[(3)].

For a certain cotorsion pair $(A, A^{\perp})$, the algebraic K-theory can be defined by Sarazola[(5)]. For a subcategory D of the category of A#H-modules, we denote by D^f the full subcategory consisting of finitely generated A#H-modules.

The main results are as follows.

<u>Theorem</u>

If any finitely generated B-module is with finite projective dimension as an B-module, the K-theory arising from a certain triple $K(qCof^f, qCof^f \cap qAcyc^f)$ is weakly equivalent to the G-theory $K(LMod(B)^f, qAcyc^f)$.

If H-module algebras A satisfy the assumption that all finitely generated A#H-modules have finite projective dimension as A#H-modules, we need to choose A not to be Frobenius.

2 Notations

Let H be a finite dimensional Hopf algebra over a field k.

A k-algebra A is called a left H-module algebra if A is a left H-module such that $h(ab)=\Sigma(h_{(1)}a)(h_{(2)}b)$ and $h1=\varepsilon(h)1$ for all a, b in A and h in H. Here, we use the Sweedler notation for Hopf algebras and ε is the counit of H.

The smash product algebra (or semidirect product) of A with H, denoted by A#H, is the vector space $A\otimes H$, whose elements are denoted by a#h instead of $a \otimes h$, with multiplication given by $(a\#h)(b\#j)=\Sigma a(h_{(1)}b)\#(h_{(2)}j)$ for a, b in A and h, j in H. The unit of A#H is 1#1. We let B = A#H.

Since the Hopf algebra H finite dimensional over a field k

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is a Frobenius algebra, so that it is self-injective. We will say that λ in H is the left integral if it satisfies $h\lambda=\varepsilon(h)\lambda$ for all h in H.

Let LMod(H) be the category of left H-modules. The associated stable module category, denoted by lmod(H), whose objects are the same as those of LMod(H) and the morphism set between two left H-modules X and Y is defined by the quotient $Hom_{lmod(H)}(X, Y)=Hom_{LMod(H)}(X, Y) / I(X, Y)$, where I(X, Y) is the space of morphisms between X and Y that factor through an injective (projective) H-module. We call those morphisms null-homotopic. We can also define the associated category lmod(B) of LMod(B), whose objects are the same as those of LMod(B) and the morphism set between two left B-modules X and Y is defined by the quotient $Hom_{lmod(B)}(X, Y)=Hom_{LMod(B)}(X, Y) / I(X, Y)$, where I(X, Y) is the space of morphisms between X and Y that factor through a B-module which is an injective (projective) H-module. Those morphisms are also called null-homotopic. We will say that a B-module M is contractible if it is a projective (injective) as a left H-module.

The shift functor T on lmod(H) and lmod(B) are given as follows, respectively: for any H-module (resp. B-module) M, let M⊆I be the inclusion of M into the injective H-module $I=M\otimes H$, given by $Id_M\otimes\lambda: M\to M\otimes H$. Here, λ is a left integral in H. Then T(M) is defined to be the cokernel $M\otimes(H / \lambda)$ of this inclusion. Similary, the inverse suspension is defined by $T^{-1}(M)=M \otimes Ker\varepsilon$, where $\varepsilon: H\to k$ is the augmentation of H. The stable category lmod(H) and lmod(B) are triangulated monoidal, respectively.

Definition

Let H be a finite-dimensional Hopf algebra over k. The morphism set of two H-modules M, N in the stable category lmod(H) is canonically isomorphic to the quotient space $Hom_k(M, N)^H / \lambda Hom_k(M, N)$. For any H-module V its space of stable invariants is defined to be the k-vector space $H(V)=V^H / \lambda V$. For a B-module V, H(V) is defined as above regarding V as an H-module.

For any two B-modules M and N, the space I(M, N) of null-homotopic morphisms in $Hom_B(M, N)$ is naturally identified with $I(M, N)=\lambda Hom_A(M, N)$, where the right-hand side is regarded as a k-subspace of $Hom_A(M, N)^H=Hom_B(M, N)$.

We define the homology group $H_0(Hom_A(M, N))$ of Hom-set by $Hom_{lmod(B)}(M, N)$.

3. A tiple

Let C be an abelian category and D a subcategory.

For objects x and y in C, we say that x is orthogonal to y if Ext(x, y)=0.

For c in C, let us denote two sets by $D^{\perp}=\{y \text{ in } C \mid Ext^1(c, y)=0$ for every c in D$\}$ and ${}^{\perp}D =\{x \text{ in } C \mid Ext^1(x, c)=0$ for every c in D$\}$.

For the categories F and D of C, a pair (F, D) is a cotorsion pair if $F^{\perp}=D$ and $D^{\perp}=F$.

A cotorsion pair has enough projective if for any N in C, these exists an exact sequence of the form $0\to L\to M\to N\to 0$ with L in D and M in F.

A cotorsion pair has enough injective if for any L in C, these exists an exact sequence of the form $0\to L\to M\to N\to 0$ with M in D and N in F.

A cotorsion pair is complete if it has both enough projective and injective.

For A#H-modules X and Y, we define n-th Ext-groups $Ext^n(X, Y)$ of X and Y to be the homology group $H_0(Hom_A(X, T^n(Y))) = Hom_{lmod(B)}(X, T^n(Y))$.

Note that $Ext^1(X, Y)$ is the biadditive functor.

A complete cotorsion pair corresponds to an abelian model structure. Now, we will summarize some classes of objects in LMod(B) arising from the projective model structure, which is given by a certain cotorsion pair defined in the paper(3).

Definition

M is said to be q-cofibrant if for any surjective q-equivalence L→N of B-modules, the induced map of k-vector spaces $Hom_B(M, L)\to Hom_B(M, N)$ is surjective.

We define a q-equivalence to be an A-module map between B-modules which induces an isomorphism on thier homology groups H_n for all n in Z. We say M is q-acyclic if $H_n(M)$ is isomorphic to zero for all n in Z.

We let qCof be the full subcategory of LMod(B) consisting of q-cofibrant B-modules and qAcyc the full subcategory consisting of q-acyclic B-modules.

The class of q-equivalences constitute a localizing class. The localization of lmod(B) via q-equivalences is denoted by $D_{A,H}$, and we say this as the derived category of left B-modules. The category $D_{A,H}$ is triangulated.

4 Lemmas for K-theory and G-theory

If we have a triangle $X\to Y\to Z\to TX$ in the stable category lmod(B), then there is a long exact sequence of homology groups as k-modules.

We will take a projective resolution of a B-modules as B-modules, which is related to homology and Ext groups.

Lemma

Let M be an q-cofibrant B-module. Let us take a surjection φ: $\oplus_i$(A#H)→M. Then, the kernel Kerφ_1 is q-equivalent to Σ^{-1}M.

Proof.

Since M is q-cofibrant, it is projective as A-modules. Therefore, the short exact sequence 0→Kerφ_1→$\oplus_i$(A#H)→M→0 of B-modules are split as A-modules. Then, we have a long exact sequence of homology groups in the stable category lmod(B).

The following proposition is an analogue of DG case.

Proposition

Let M be a B-module. Then, M is q-cofibrant and its projective resolution has finite end if and only if there exists i≧0 such that Ext^i(M, N)=0 for any q-acyclic B-module N and H_n(M)=0 for sufficiently large n≧i.

Proof.

Only if direction: If we have a projective resolution as B-modules and it finitely ends, the final kernel Kerφ_i is B-projective by assumption and q-equivalent to T^{-i}M by the previous lemma. Therefore Ext^i(M, N)=Ext(Kerφ_i, N)=Ext(T^{-i}M, N)=0 for any q-acyclic B-module N since Kerφ_i is q-cofibrant. For n≦i, H^{-n}(M)=H_0(T^{-n}M)=H_0(Kerφ_i)=0.

If direction: Take a projective resolution as B-modules and take the functorial q-cofibrant replacement. Then, by the previous lemma, is q-equivalent to T^{-i}M. By assumption, T^{-i}M is q-cofibrant and T^{-n}M is q-acyclic for i.

By Tamaki and the author's paper(3), q-cofibrant and q-acyclic is B-projective, so Kerφ_i is B-projective, hence the resolution finitely ends.

We will apply the following resolution theorem for the K-theory of cotorsion pair or the K-theory of relative category Theorem 7.2 of Sarazola(5) to our case.

Theorem

Let E be an exact category, and C, Z two full subcategories of E such that C is closed under kernels of admissible epimorphisms and is part of a complete cotorsion pair (C, $C^\perp$), with $C^\perp$⊆Z, and such that Z∩C has 2-out-of-3 for short exact sequences in C.

Let P be a full subcategory of C such that P is closed under extensions and kernels of admissible epimorphisms in C. In addition, assume that the cotorsion pair (P, $P^\perp$) (defined with respect to Ext_P is complete, and that $P^\perp$⊆Z. If every object M in C admits a finite P-resolution

$0\to P_n\to\ldots\to P_1\to P_0\to M\to 0$,

then, we have a homotopy equivalence between K(P, $w_{Z\cap P}$) K(C, w_Z).

5 Conclusion

By applying the previous theorem, we have the following.

Corollary

If any finitely generated B-module is with finite projective dimension as an B-module, the K-theory K($qCof^f$,$qCof^f$∩$qAcyc^f$) is homotopy equivalent to the G-theory K($LMod(B)^f$, $qAcyc^f$).

Now, we dispute the assumption with respect to the above Corollary.

We have the following known results.

Proposition

(Eilenberg-Nakayama Theorem(1)):Let k be a commutative ring. Let R be a left Noetherian k-algebra which is left self-injective. Then, the global dimension of R is 0 or infinity.

(Auslander-Buchsbaum Theorem):For a Noethrian regular local ring, let M be a finitely generated module with finite projective dimension over an Artinian local ring, then it is finitely generated projective.

Remark

For a left Artinian ring R, its global dimension Sup{Projdim(M) | M : simple left R module} can be described as Projdim(J(R))+1.

Remark

In general, a module with finite projective (resp. injective) dimension may not has finite injective (resp. projective) dimension. Non-regular Gorenstein rings are example.

Now, readers may wonder if the assumption, for an H-module algebras A, that all finitely generated A#H-modules have finite projective dimension as A#H-modules, and how many H-module algebras A satisfy the assumption. In fact, any left module of finite projective dimension over a Frobenius algebra has left global dimension 0 and an associative ring has left global dimension 0 if and only if it is semisimple. Now, the assumption says that all finitely generated modules had projective dimension 0. Note that A#H is finitely generated as A#H-modules. Also, A#H is Frobenius if and only if A is Frobenius. Thus, under our assumption, A is not to be Frobenius.

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